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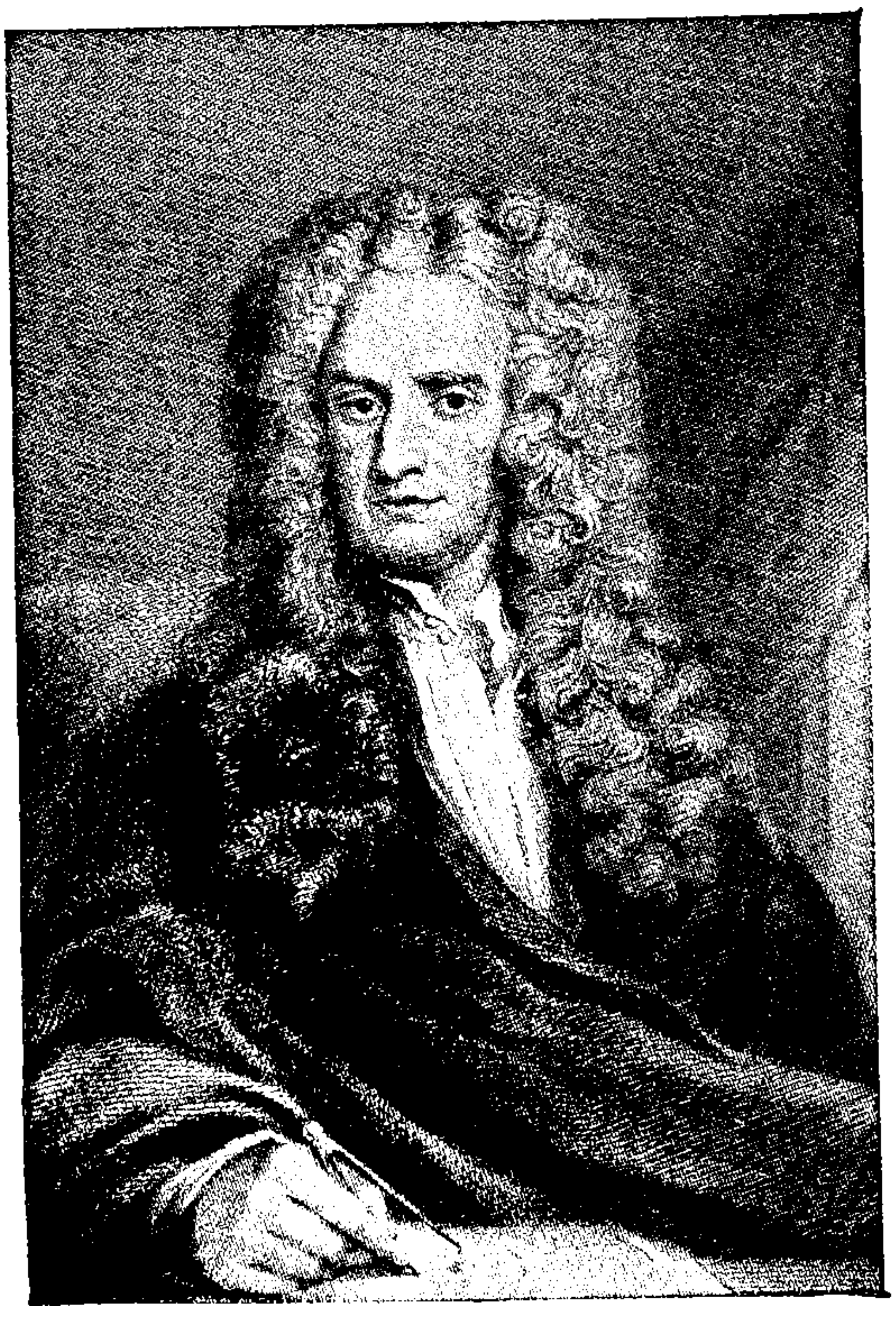

NEWTON

GAUSS

DURELL'S

SCHOOL ALGEBRA

BY

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PREFACE

THE main object in writing this SCHOOL ALGEBRA has been to *simplify principles and give them interest,* by showing more plainly, if possible, than has been done heretofore, the *practical or common-sense reason* for each step or process. For instance, at the outset it is shown that new symbols are introduced into algebra not arbitrarily, but because of definite advantages in representing numbers. Each successive process is taken up for the sake of the economy or new power which it gives as compared with previous processes.

This treatment should not only make each principle clearer to the pupil, but should give increased unity to the subject as a whole. We believe also that this treatment of algebra is better adapted to the practical American spirit, and gives the study of the subject a larger educational value.

Among the special features of this SCHOOL ALGEBRA, the following may be mentioned:

A large number of *written problems* are given in the early part of the book, and these are grouped in types which correspond in a measure to the groups used in treating original exercises in the author's GEOMETRY.

Many *informational facts* are used in the written problems. The central and permanent numerical facts in various departments of knowledge have been collected and tabulated on pages 496–504 for use in making problems. Similarly the most important formulas in arithmetic, geometry, physics, and engineering have been tabulated for use by teacher and pupil (pp. 496, 497).

The *self-activity of the pupil* is aroused by examples which require the pupil to invent and solve problems of a specified kind, material for such examples being made available in the tables of formulas and numerical facts.

Many of the examples in the book require a frequent *review of the principles of arithmetic,* as of decimal fractions and percentage.

Numerous and thorough *reviews* of the portion of the Algebra already studied are also called for. A unique feature is the series of spiral reviews of the preceding part of the book by means of examples at the end of Exercises. *Oral work* is called for in like manner and is also emphasized in special important Exercises.

The *utilities in symbolism in general,* apart from technical algebra, are brought out in a special Exercise (pp. 249, 250) and thus the direct practical value of the study of algebra is much broadened.

The *history of algebra* is discussed in Chapter XXVI, and questions on this chapter are inserted in appropriate places in the text.

The author wishes to express his indebtedness to Professor William Betz of the East High School, Rochester, New York, and to Dr. Henry A. Converse of the Polytechnic Institute, Baltimore, Maryland, for important aid in preparing the book. He is indebted also to *School Science and Mathematics* and the *Mathematics Teacher* for a few of the problems.

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SCHOOL ALGEBRA

CHAPTER I

ALGEBRAIC SYMBOLS

1. The Use of Letters.

Ex. Walter and Harold made \$27 by gardening one summer. Walter, who was older and stronger, received a double share of the profits. How much did each receive?

SOLUTION WITHOUT THE AID OF X

1 share = Harold's part of the profits
2 shares = Walter's part of the profits
1 share + 2 shares = \$27
3 shares = \$27
1 share = \$9, *Harold's part*
2 shares = \$18, *Walter's part*

SOLUTION BY AID OF X

Let x = Harold's part of the profits
Then $2x$ = Walter's part of the profits
Hence $x + 2x = \$27$
$3x = \$27$
$x = \$9$, *Harold's part*
$2x = \$18$, *Walter's part*

We see that by use of the letter x the solution is much shortened.

2. Algebra is that branch of mathematics which treats of number by the extended use of symbols.

Later algebra comes to have a wider meaning.
Algebra may also be briefly described as generalized arithmetic.

3. Utility of Algebra. A more extended use of symbols than is practiced in arithmetic (1) shortens the work of solving problems; (2) enables us to solve problems which we could not otherwise solve; and (3) gives other advantages which will become evident as we proceed (see Art. 143 and Exercise 76, p. 249).

EXERCISE 1

(Problems of **Type I**, i. e. of the form $\mathbf{x + ax = b}$.)

1. Two boys together catch 84 fish. If the boy who owns the boat which they use, receives twice as many fish as the other boy, how many fish does each boy receive?

2. A man left $12,000 to his son and daughter. To his daughter, who had taken care of him in his old age, he left a double share. What did each receive?

3. A man and boy by working a garden one summer made $128.80. If the man received a share of the profits three times as large as the share received by the boy, how much did each receive?

4. Two boys together gathered 1 bu. 4 qt. of hickory nuts. If the boy who climbed the trees received a double share, how many quarts did each receive?

5. Make up and work a similar example concerning two boys who gathered chestnuts.

6. Two girls made $18.60 by sewing. The girl who supplied the thread and machine received twice as much as the other girl. How much did each make?

7. Make up and work a similar example concerning two girls who kept a refreshment stand.

8. Solve Ex. 1 without the use of x (see Art. 1). How much of the labor of writing out the solution is saved by the use of x? Is there any other advantage in the use of x in solving a problem?

9. The total cotton crop of the world in a certain year was 15,000,000 bales, and the United States in that year produced three times as much as all the rest of the world. How many bales of cotton did the United States produce?

10. A farm is worked on shares. As the tenant supplied the tools and fertilizers, he received twice as large a share of the profits as the owner of the farm. If the profits for one year are $6000, how much does the tenant receive? The owner?

11. If the sum of the areas of New York and Massachusetts is 57,400 sq. mi. approximately, and New York is 6 times as large as Massachusetts, what is the area of each state?

12. One number is 5 times as large as another and the sum of the numbers is 240. Find the numbers.

13. One number is twice as large as another and the sum of the numbers is 7.26. Find the numbers.

14. One fraction is three times as large as another and their sum is $\frac{1}{3}$. Find the fractions.

15. One number is 4 times as large as another and their sum is .0045. Find the numbers.

16. Separate $120 into two parts such that one part is three times as large as the other.

SUG. Let x = the smaller part.

17. Separate $5\frac{1}{3}$ into two parts such that one part is 7 times as large as the other.

18. Make up and work an example similar to Ex. 11. Also one similar to Ex. 15. To Ex. 16.

Material for examples may be obtained from the lists of Important Numerical Facts given on pp. 498–504.

19. To look well, the middle part of a steeple should be twice as high as the lowest part, and the top part 8 times as high as the lowest part. If a steeple is to be 132 ft. high, how high should each part be?

20. A man wants to save $6000 in three years. If he is to save twice as much the second year as the first, and three times as much the third year as the first, how much must he save each year?

21. A girl has $42 to spend for a hat, coat, and suit. She wants to spend twice as much for her coat as for her hat, and three times as much for her suit as for her hat. How much does she spend for each?

22. A man bequeathed $84,000 to his niece, daughter, and wife. If the daughter received twice as much as the niece, and the wife four times as much as the niece, how much did each receive?

23. A certain kind of concrete contains twice as much sand as cement and 5 times as much gravel as cement. How many cubic feet of each of these materials are there in 1000 cu. yd. of concrete?

24. Make up and work a similar example for yourself where the materials in the concrete are as 1, 2, 4.

25. In a certain kind of fertilizer the weight of the nitrate of soda equals that of the ground bone, and the weight of the potash is twice as great as that of the ground bone. How many pounds of each of the materials are there in a ton of fertilizer?

26. If the amount of potash in a given kind of glass is 5 times as great as the amount of lime, and the amount of sand 3 times as great as the amount of potash, how many pounds of each will there be in 4000 lb. of glass?

27. The railroad fare for two adults and a boy traveling for half fare was \$49.50. What was the fare for each person?

SUG. Let x = the smallest of the fares.

28. Separate 120 into three parts, such that the second part is twice as large as the first, and the third part three times as large as the first.

29. Separate 120 into three parts which shall be as 1, 2, 3.

30. Separate .0372 into three parts in like manner. Also $\frac{5}{16}$.

31. Separate 240 into four parts which shall be as 1, 1, 2, 4.

32. Separate \$1800 into three parts, such that the second is three times as large as the first, and the third 5 times as large as the second.

33. In one kind of concrete the parts of cement, sand, and gravel are as 1, 2, and 4; in another kind three parts are as 1, 2, and 5. How many more pounds of cement are needed in a ton of one than of the other?

34. How many of the examples in this Exercise can you work at sight?

35. To get the greatest possible benefit from the use of letters to represent numbers, we now make further definitions and rules.

4. Three Classes of Symbols. Three principal kinds of symbols are used in algebra: (1) Symbols of *quantity,* (2) Symbols of *operation,* and (3) Symbols of *relation.*

5. Symbols for Known Quantities. Known quantities are represented in arithmetic by figures; as 2, 3, 27. They are represented in the same way in algebra, but also in another more general way, viz.: by letters; as by a, b, c.

The advantages in the use of letters to represent known numbers are: (1) letters are brief to write; and (2) a letter may stand for any known number, and thus by the use of letters we obtain results which are true for all numbers. See Exs. 34–40, p. 97.

6. Symbols for Unknown Quantities. Unknown quantities in algebra are usually denoted by the last letters of the alphabet; as x, y, z, u, v, etc.

The advantages in the use of distinct symbols for unknown quantities are numerous and will be gradually realized as we proceed. Some of these advantages are stated in Art. 3. See also Art. 143.

7. The Signs $+$, $-$, $\times$, $\div$, and $=$ are used in algebra, as in arithmetic, to denote addition, subtraction, multiplication, division, and equality respectively.

In algebra, multiplication is also denoted by a dot placed between the two quantities multiplied, or by placing the quantities side by side without any intervening symbol.

Thus, instead of $a \times b$, we may write $a \cdot b$ or ab.

8. Signs of Aggregation. The parenthesis sign, ( ), is used, as in arithmetic, to indicate that all the quantities inclosed by it are to be treated as a single quantity; that is, subjected to the same operation.

Thus, $5(2a - b + c)$ means that the quantities inside the parenthesis, viz. $2a$, $-b$, and $+c$, are each to be multiplied by 5.

Again, $(a + 2b)(a + 2b + c)$ means that the sum of the quantities in the first parenthesis is to be multiplied by the sum of those in the second parenthesis.

Instead of the parenthesis, to prevent confusion, the following signs are sometimes used: the brackets [ ], the braces $\{\}$, and the vinculum ——.

9. **The Sign of Continuation** is . . . . This sign is read "and so on" or "and so on to."

Thus, 1, 3, 5, 7, ... is read "1, 3, 5, 7 and so on."
But 1, 3, 5, 7, ... 19 is read "1, 3, 5, 7 and so on to 19."

10. **The Sign of Deduction** is $\therefore$ and it is read "therefore" or "hence."

This sign is used to show the relation between succeeding propositions.

EXERCISE 2

Express in words:

1. $5 + a$.
2. $d - a$.
3. $a \div b$.
4. ad.
5. $2a + 3b$.
6. $\frac{c}{a} - \frac{d}{b}$.
7. $5b - a$.
8. $2a + 3c$.
9. $cd - ab$.
10. $7(a + b)$.
11. $7(a - b)$.
12. $\frac{5a + b}{x + y}$.
13. $a + b \div 3$.
14. $4 + 5(a + b)$.
15. $(a + b)(x - y)$.
16. $2a + 3b - 5c$.
17. $a \div (x + y)$.
18. $\frac{a + b}{5} + \frac{c}{d}$.

19. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the value of the combinations of symbols in Exs. 1–10.

20. Make and read an example similar to Ex. 5. To Ex. 10. Ex. 14.

Express in symbols:

21. x plus 3. The sum of x and 3. The number which exceeds x by 3.

22. x diminished by 3. The number 3 less than x.

23. Two times a plus three times b.

24. The sum of 4 and of 5 times x.

25. One third of the sum of a and b.

Answer the following in algebraic language:

26. If a boy has a cents and earns 10 cents, how many cents will he then have?

27. How many, if he has a cents and earns b cents? How many, if he then spends c cents?

28. Walter has x marbles and his brother has 10 more than Walter. How many marbles has his brother?

29. Walter has b marbles and his brother has 5 more than twice Walter's marbles. How many has his brother?

30. If Mary is a years old now, how old will she be in 3 years? In 5 years? In x years?

31. What is the next larger number than 5? Than x? n? $x + 1$? $x + 2$? $n - 1$? $x - 2$?

32. What is the next larger even number than 6? Than $2y$? $2x$? $2n + 2$?

33. Taking x as the smallest number, write two consecutive numbers. Three consecutive numbers. Four. Five.

(The following problems are mainly of **Type II**, i. e. of the form $\mathbf{x + x + a = b}$.)

34. If there are 214 pupils in our school, and the number of girls exceeds the number of boys by 8, how many boys and how many girls are there?

$$\begin{array}{lrcl}
\text{Let} & x & = & \text{the number of boys} \\
\text{Then} & x + 8 & = & \text{the number of girls} \\
\text{Hence} & x + x + 8 & = & 214 \\
\text{Or} & 2x + 8 & = & 214 \\
\text{Subtracting 8 from the} & -8 & & -8 \\ \hline
\text{equals gives} & 2x & = & 206 \\
 & x & = & 103, \textit{ the number of boys} \\
 & x + 8 & = & 111, \textit{ the number of girls}
\end{array}$$

35. Walter and his brother together had 60 marbles, and his brother had 10 more than Walter. How many marbles had each boy?

36. Make up and work an example similar to Ex. 35.

37. At New York on Dec. 21, the night is 5 hr. 32 min. longer than the day. Find the length of the day.

38. Separate $28\frac{1}{2}$ into two parts such that one shall exceed the other by $2\frac{3}{4}$.

39. A baseball nine has played 62 games and won 8 more games than it has lost. How many games has it won?

40. In a certain election 12,784 votes were cast. If the successful candidate had a majority of 1732, how many votes did he receive?

41. Make up and work an example similar to Ex. 40.

42. The sum of two consecutive numbers is 15. Find the numbers.

43. The sum of three consecutive numbers is 33. Find the numbers.

44. If 112,216 sq. mi. are added to 24 times the area of the British Isles, the result will be 3,025,600 sq. mi. (the area of the United States). Find the area of the British Isles.

45. Twice the height of Mt. Washington with 1567 ft. added equals the height of Pike's Peak, or 14,137 ft. Find the height of Mt. Washington.

46. How many of the examples in this Exercise can you work at sight?

47. Which of the symbols mentioned in Arts. 6–10 are symbols of quantity? Of operation? Of relation?

48. Make up and work an example similar to Ex. 44. To Ex. 45.

DEFINITIONS AND PRINCIPLES

11. The term **Factors** has the same meaning in algebra as in arithmetic; that is, the factors of a number are the numbers which, multiplied together, produce the given number.

For example, the factors of 14 are 7 and 2; the factors of abc are a, b, and c.

12. Coefficients. A numerical factor, if it occurs in a product, is written first and is called a *coefficient.* Hence,

A **coefficient** is a number prefixed to a quantity to show how many times the given quantity is taken.

For example, in $5xy$, 5 is the coefficient.

When the coefficient is 1, the 1 is not written, but is understood.

Thus, xy means $1xy$.

The following enlarged definition of cœfficient is often used. In the product of several factors, the *coefficient* of any factor, or factors, is the product of the remaining factors.

Thus, in $5abxy$, the coefficient of y is $5abx$; of xy, is $5ab$; of ab is $5xy$. What is the coefficient of b? Of a? x? $5a$? 5?

A **numerical coefficient** is a coefficient composed only of figures; as 15 in $15ab$.

A **literal coefficient** is a coefficient composed only of letters; as ab in abx.

What, then, is a *mixed coefficient?* Give an example of one.

13. Power and Exponent are used in the same sense in algebra as in arithmetic.

A **power** is the product of equal factors.

A power is expressed briefly by the use of an exponent.

An **exponent** is a small figure or letter written above and to the right of a quantity to indicate how many times the quantity is taken as a factor.

Thus, for $xxxx$, or four x's multiplied together, we write x^4, the exponent in this case being 4. The expression is read "x to the fourth power."

When the exponent is unity, it is omitted. Thus, x is used instead of x^1, and means x to the first power.

A power is composed of two parts: (1) the base (i. e. one of the equal factors); and (2) the exponent.

Thus, in the power a^3, the base is a and the exponent is 3.

14. Root and Radical Sign have the same meaning in algebra as in arithmetic.

A **root** of a number is one of the equal factors which, when multiplied together, produce the given number.

The **square root** of a number is one of *two* equal factors which, multiplied together, produce the given number. What is the *cube root* of a number? The fifth root?

Thus, 4 is the cube root of 64, and a of a^3.

The **radical sign** is $\sqrt{}$, and means that the root of the quantity following it is to be found. The degree of the root is indicated by a small figure placed above the radical sign.

The number denoting the degree of a root is the *index* of the root. For the square root, the figure or index of the root is omitted.

Thus, $\sqrt{9}$ means "square root of 9."
$\sqrt[3]{a}$ means "cube root of a."

15. Aids in Solving Problems; Axioms In solving problems like those given in Exercise 1 and the latter part of Exercise 2, certain principles are often important aids in discovering the relations used and simplifying them.

The most important of these principles are as follows:

1. *The whole is equal to the sum of its parts.*
2. *Things equal to the same things, or equal things, are equal to each other.*
3. *If equals are added to equals, the results are equal.*
4. *If equals are subtracted from equals, the results are equal.*
5. *If equals are multiplied by equals, the results are equal.*
6. *If equals are divided by equals, the results are equal.*
7. *Like powers, or like roots, of equals are equal.*

These principles are sometimes called *axioms*.

EXERCISE 3

Write in words:

1. $5b^3$.

2. b^3c^2.

3. $3b^2c^3$.

4. 3^b.

5. $\frac{1}{6}b^2c^3$.

6. $2a^2 + 3b^3$.

7. $\dfrac{a^2 + b^2}{3}$.

8. $(a + b)^2$.

9. $5(b - a)^2$.

10. $a + (b + c)^2$.

11. $\dfrac{9a^2}{c} - \dfrac{3b^2}{a}$.

12. $\sqrt{a} + \sqrt[3]{b}$.

13. $\sqrt{7a + b}$.

14. $7\sqrt[5]{6a + b}$.

15. $\dfrac{\sqrt{a}}{5} - \dfrac{\sqrt{b}}{4}$.

16. If $a = 1$, $b = 2$, and $c = 3$, find the value of the combinations of symbols in Exs. 1–8.

17. If $a = 4$, $b = 8$, and $c = 3$, find the value of the expressions in Exs. 9–12.

Write in symbols:

18. The square of the sum of a and b. Of $2a$ minus $3b$.

19. The cube root of the sum of a and b.

20. x plus x increased by 4 equals 14.

21. x plus twice x plus x increased by 3 equals 108.

22. Make up and work an example similar to Ex. 18. To Ex. 20.

23. Reduce to its simplest form $b + b + b + b + b$. Also $b \times b \times b \times b \times b$.

If $b = 2$, what is the value of each of these results?

24. Make up and work an example similar to Ex. 23.

25. Reduce $3aaa + 7bbbb - 5cccccc$ to its simplest form. How many more symbols are used in the long form than in the short form?

26. Find the value of 2^n when $n = 1$. Also when $n = 2$. 3. 5. 7.

27. Find the value of a^n, when $a = 3$ and $n = 4$. Also when $a = 5$ and $n = 3$.

28. Express the number of your great-grandparents as a power of 2.

(The following are miscellaneous problems of **Types I** and **II**.)

29. A man and boy together spade up a garden containing 6000 sq. ft. If the man spades four times as much ground as the boy, how much does the boy spade?

30. Two boys earn $38 by taking passengers on a motor boat. If the boy who owns the boat receives $10 more than the other boy, how much does each receive?

31. A certain macadam road cost $1800, of which the county paid twice as much as the state, and the township the same amount as the county. How much did each pay?

32. The top of the Statue of Liberty in New York Harbor is 306 ft. above the surface of the water. If the altitude of the pedestal is 4 ft. greater than the height of the statue, how high is each?

33. In a certain kind of gunpowder the weight of the charcoal equals that of the sulphur, and the amount of niter equals the charcoal and sulphur combined. How many pounds of each substance are needed to make a ton of gunpowder?

34. In a certain year in the United States 200,000,000 bushels plus three times the number of bushels in the wheat crop equaled the corn crop, or 2,600,000,000 bushels. How many bushels were in the wheat crop?

35. Point out the problems among Exs. 29–34 which belong to Type I. Also those which belong to Type II.

36. Make up and work an example similar to Ex. 29. To Ex. 31.

37. How many of the examples in this Exercise can you work at sight?

ALGEBRAIC EXPRESSIONS

16. **An Algebraic Expression** is an algebraic symbol or combination of symbols representing some quantity; as $5x^2y - 6ab + 7\sqrt[3]{ax}$.

17. **A Term** is a part of an algebraic expression which does not contain a plus or minus sign. (Signs occurring inside a parenthesis are not considered in fixing the terms.)

Ex. 1. $5x^2y - 6ab + 7\sqrt[3]{ax}$.

This algebraic expression contains three terms: viz. $5x^2y$, $-6ab$, and $7\sqrt[3]{ax}$.

Ex. 2. $5x + a \div b + c$.

This expression also contains three terms: $5x$, $a \div b$, and c.

Ex. 3. $7ax^2 + 5(a + b) - c^3$.

Since the parenthesis, $(a + b)$, is treated as a single quantity, three terms occur in this expression: $7ax^2$, $5(a + b)$, and $-c^3$.

18. **A Monomial** is an algebraic expression of only one term; as $5x^2y$ or c.

19. **A Polynomial** is an algebraic expression containing more than one term; as $3ab - c + 2x + 5y^2$.

A monomial is sometimes called a *simple expression*, and a polynomial a *compound expression*.

20. **A Binomial** is an algebraic expression of two terms; as $2a - 3b$.

A Trinomial is an algebraic expression of three terms; as $2a - 3b + 5c$.

EVALUATION OF ALGEBRAIC EXPRESSIONS

21. The Order of Operation in obtaining numerical values is the same in algebra as in arithmetic.

I. In a series of operations involving addition, subtraction, multiplication, division, and root extraction, *the multiplications, divisions, and root extractions are to be performed before any of the additions and subtractions.*

Ex. 1. Find the value of $4 + 12 \times 3$.

$$4 + 12 \times 3 = 4 + 36 = 40 \textit{ Ans.}$$

(hence $4 + 12 \times 3$ does not equal 16×3, etc.)

Ex. 2. What is the value of $60 - 8 \div 2 + 3 \times 7$?

$$60 - 8 \div 2 + 3 \times 7 = 60 - 4 + 21 = 7 \textit{ Ans.}$$

II. If a given expression contains one or more parentheses (or other signs of aggregation), *each parenthesis is to be reduced to a single number before the operations of the expression as a whole are to be performed.*

Ex. 1. $5 + 4(6 - 2) = 5 + 4 \times 4 = 5 + 16 = 21$ *Ans.*

(hence $5 + 4(6 - 2)$ does not equal $9(6 - 2)$ or 9×4, etc.)

Note that in an expression like $\sqrt{16 + 9}$ the bar above the $16 + 9$ is a vinculum, or sign of aggregation.

Ex. 2. $\sqrt{16 + 9} = \sqrt{25} = 5$ *Ans.*

(hence $\sqrt{16 + 9}$ does not equal $\sqrt{16} + \sqrt{9}$, etc.)

22. The Numerical Value of an Algebraic Expression is obtained thus:

Substitute for each letter in the expression the number which the letter stands for;

Perform the operations indicated.

Thus, if $a = 1$, $b = 2$, $c = 3$:

Ex. 1. Find the numerical value of $7ab - c^2$.

$$\begin{aligned} 7ab - c^2 &= 7 \times 1 \times 2 - 3^2 \\ &= 14 - 9 \\ &= 5 \; \textit{Ans.} \end{aligned}$$

Ex. 2. Find numerical value of $\frac{9b}{c} + 5ab^2 - 7(a^3 + 2b)^2 + 3c^2$.

The given expression

$$\begin{aligned} &= \frac{9 \times 2}{3} + 5 \times 1 \times 2^2 - 7(1^3 + 2 \times 2)^2 + 3 \times 3^2 \\ &= 3 \times 2 - 5 \times 4 + 7(1 + 4)^2 + 3 \times 9 \\ &= 6 - 20 + 175 + 27 \\ &= 188 \; \textit{Ans.} \end{aligned}$$

EXERCISE 4

In each of the following examples, state the order of operations before working the example. Wherever possible, use cancellation. When $a = 5$, $b = 3$, $c = 1$, and $x = 6$, find the numerical value of

1. $2 + 3a$.
2. $x - 2c$.
3. $4b - 2x$.
4. $a + 3x$.
5. $5a - 3x$.
6. $3(a + c)$.
7. $a + 3c - x$.
8. $5x - 2b + a$.
9. $a + x \div b - c$.
10. $5\,x \div b - c$.
11. $3b - x$.
12. $2x - 4bc$.
13. $a^3 - bx^2$.
14. $3(a + c)$.
15. $x(a - b)$.
16. $4(a - 3c)^2$.
17. $2x(2a - 3b)^2$.
18. $3 + 2(x - a)$.
19. $5x - 3(2b + c)$.
20. $2(x^2 - a^2) + 3ac$.
21. $3x(x - 3)^2 - 9x$.
22. $(x - 1)(x - 3) + x(x - a)$.
23. $3(2x - 5c) - a(2b^2 - 3x)$.
24. $(5b + x)(x - b + a - 5c^2)$.

25. $\dfrac{a+7c}{x}$.

26. $\dfrac{3x^2-b}{a+2c}$.

27. $\dfrac{5a^2}{x-1}+\dfrac{3c}{b}$.

28. $\dfrac{(x-1)(b+1)(5c-b)}{abx}$.

29. $\left(\dfrac{3a^2b}{5x}\right)\left(\dfrac{2b}{x-1}\right)\left(\dfrac{4c-1}{3b}\right)$.

If $a=\frac{1}{2}$, $b=\frac{2}{3}$, $x=2$, $y=\frac{3}{5}$, find the value of

30. $6a$.

31. by.

32. abx.

33. a^2x^2.

34. $3ab^2$.

35. $x-2b$.

36. $2x+5y$.

37. $6ab-b^2y$.

38. $b(10y-3b)$.

39. $3x(4a+3b)$.

40. $ax+5x(3b-y)$.

41. $3a+a(3x-10y)$.

42. $5x-3(by-ab)$.

43. $5x(by-a^2)-bx$.

44. $6(a+b)^2+10(y-a)^2$.

45. $x+\sqrt{8a}$.

46. $\sqrt{8a}+\sqrt{3b}$.

47. $5y-\sqrt{9ax}$.

48. Does $x^2+x=12$, if $x=2$? If $x=3$? 4? 5? 1?

49. Does $3x^2-4x=4$, if $x=1$? If $x=2$? 3? $\frac{2}{3}$? 0?

50. Does $x^2-5x+6=0$, if $x=1$? If $x=2$? 3? 4? 5?

51. Does $x^2-\frac{7}{3}x-2=0$, if $x=1$? If $x=2$? 3? 4? $\frac{1}{2}$?

52. Show that $(a-2b)^2=a^2-4ab+4b^2$, when $a=3$ and $b=1$.

53. That $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$, when $a=2$ and $b=1$.

54. Find the value of $2x^2$ when $x=1$. When $x=2$. 5. $\frac{1}{2}$. 1.5.

SUG. The results may be conveniently arranged as in the following tabulation:

x	$2x^2$
1	2
2	8
5	50
$\frac{1}{2}$	$\frac{1}{2}$
1.5	4.5

Find the value of each of the following and tabulate results:

55. $2x+1$, when $x=1$. When $x=2$. 3. 5. $\frac{1}{2}$. $\frac{1}{4}$. 1.5.

56. $x^2 + 2$, when $x = 1$. When $x = 2$. 3. $\frac{1}{2}$. 1. 5.

57. $x(x + 1)$, when $x = 1$. When $x = 2$. 3. .2. $\frac{1}{2}$. $\frac{1}{4}$.

58. In Exs. 4–10 state which of the expressions used are monomials. Also which are binomials. Trinomials. State the same for Exs. 35–40.

EXERCISE 5

1. If $A = lw$, find the value of A when $l = 12$ and $w = 5\frac{1}{2}$. Also when $l = 10.4$ and $w = 5.8$.

Do you know what use is made of the formula $A = lw$ in arithmetic in finding areas?

2. If $V = lwh$, find V when $l = 12$, $w = 5$, and $h = 3$. Also when $l = 10.4$, $w = 5.8$, and $h = 3.05$.

Do you know what use is made of the formula $V = lwh$ in arithmetic in finding volumes?

3. If $p = br$, find p when $b = 350$ and $r = 1.07$. Also when $b = 7.68$ and $r = .045$. Also when $b = 84{,}000$ and $r = .00\frac{1}{8}$.

What does the formula $p = br$ mean in arithmetic in connection with the subject of percentage?

4. If $i = prt$, find i when $p = \$300$, $r = .05$, and $t = 2\frac{1}{2}$. Also when $p = \$9327.50$, $r = .06$, and $t = 3\frac{2}{3}$.

What is the meaning in arithmetic of the formula $i = prt$?

5. If $A = \pi R^2$, find the value of A when $\pi = 3.1416$ and $R = 10$.

Do you know of any use that is made of the formula $A = \pi R^2$ in arithmetic?

6. If $h = \sqrt{a^2 + b^2}$, find the value of h when $a = 8$ and $b = 6$.

Do you know of any use that is made of the formula $h = \sqrt{a^2 + b^2}$ in arithmetic?

7. If $s = \frac{1}{2}gt^2$, find s when $g = 32.16$ and $t = 4$. Also when $g = 32.16$ and $t = 2\frac{1}{2}$.

Can you find out the meaning of the formula $s = \frac{1}{2}gt^2$?

8. A stone dropped from the top of a precipice reaches its base in 5 seconds. How high is the precipice?

9. If $C = \frac{5}{9}(F - 32)$, find C when $F = 95°$. Also when $F = 100°$.

Do you know the meaning of the formula used in this example?

10. If iron melts at a temperature of 2700° F., at what temperature does it melt on the centigrade scale?

11. If $A = \pi R^2 - \pi r^2$, $\pi = 3.1416$, $R = 13$, and $r = 12$, find A in the shortest way.

12. If 1 orange costs 3 cents, how many oranges can be bought for 12 cents? For x cents? For $x + y$ cents?

13. If 1 orange costs a cents, how many oranges can be bought for 25 cents? For x cents? For $x + y$ cents?

14. If 1 acre of land costs x dollars, what will one half an acre cost? $\frac{2}{3}$ of an acre? $\frac{3}{4}$ of an acre?

(The following problems are variations of **Type I.**)

15. If a 12-year-old boy and a 16-year-old boy together earn \$48 in mowing lawns, and the younger boy receives only half as much as the other, how much does each boy receive?

Let x = no. dollars received by 16-year-old boy
Then $\frac{1}{2}x$ = no. dollars received by 12-year-old boy
Hence $x + \frac{1}{2}x = \$48$
or $\frac{3}{2}x = \$48$
Multiplying these equal numbers by 2 (Art. 15, 5)
$3x = \$96$
Dividing equals by 3 (Art. 15, 6)
$x = \$32$, *share of older boy*
$\frac{1}{2}x = \$16$, *share of younger boy*

16. A man left $24,000 to his son and daughter. As his daughter had cared for him in his old age, he left his son only $\frac{2}{3}$ as much as he left his daughter. How much did each receive?

17. A man and boy together made $124.80 by working a garden one summer. If the boy received $\frac{1}{3}$ as much as the man, how much did he receive?

18. A farm is worked on shares. As the owner of the farm supplies the tools and fertilizers, the tenant receives only $\frac{3}{4}$ as large a share of the profits as the owner. If the profits for one year are $4410, how much does each receive?

19. Two men manage a store, and as one of them owns the building, the other receives only $\frac{3}{5}$ as large a share of the profits as the owner of the store. If the profits for one year are $6600, what does each receive?

20. Separate 126 into two parts such that one of them is $\frac{1}{5}$ as large as the other. $\frac{2}{5}$ as large.

21. Separate .028 in the same manner as in Ex. 20.

22. A macadam road cost $18,000. The county paid $\frac{1}{2}$ as much of the cost as the township, and the state paid $\frac{1}{6}$ as much as the township. How much did each pay?

23. A certain kind of concrete contained $\frac{1}{2}$ as much sand as gravel and $\frac{1}{2}$ as much cement as sand. How many pounds of each material were there in $1\frac{3}{4}$ tons of concrete?

24. Make up and work an example similar to Ex. 16. To Ex. 20.

25. How many of the examples in this Exercise can you work at sight?

CHAPTER II

NEGATIVE NUMBERS

23. Positive and Negative Quantity. *Negative* quantity is quantity exactly opposite in quality or condition to quantity taken as *positive.*

If distance east of a certain point is taken as positive, distance west of that point is called negative.

If north latitude is positive, south latitude is negative.

If temperature above zero is taken as positive, temperature below zero is negative.

If in business matters a man's assets are his positive possessions, his debts are negative quantity.

Positive and negative quantity are distinguished by the signs + and − placed before them.

Thus, \$50 assets are denoted by + \$50, and \$30 debts by − \$30. We denote 12° above zero by + 12°, and 10° below zero by − 10°.

The use of the signs + and − for this purpose, as well as to indicate the operations of addition and subtraction, will be explained in Art. 26.

24. Algebraic Numbers is a general name for both positive and negative numbers.

The **absolute value** of a number is the value of the number considered without regard to its sign.

Thus, if one man travels 5 miles east and another man travels 5 miles west, the absolute distance traveled by the two men is the same, viz.: 5 miles. The two distances traveled, however, are different *algebraic numbers*, one distance being + 5 miles and the other distance being − 5 miles.

In general the absolute value of both + 5 and − 5 is 5; and of both $+a$ and $-a$ is a.

25. The Utility of Negative Number lies in the fact that the use of negative number enables us to use two opposite or contrasted kinds of quantity in working a given problem.

Also by the use of negative quantity we are often able to choose an advantageous starting point in solving a problem.

The full meaning of these utilities and other advantages in the use of negative quantity will appear as we advance in the study of algebra.

EXERCISE 6

1. What is meant by a temperature of $-8°$? By a latitude of $-23°$? By the date -776? (Dates after the birth of Christ are taken as positive.)

2. If the temperature was 17° at noon and $-8°$ at midnight, how many degrees did it fall?

3. If in a given time the temperature should fall from $-5°$ to $-12°$, how many degrees would it fall?

4. If the temperature were 15° at a given time, what would it become after a fall of 10°? Of 28°? $-15°$?

5. If the temperature were $-8°$ at a given time, what would it become after a rise of 4°? Of 15°? $-8°$?

6. Make up and work an example similar to Ex. 3. Also to Ex. 5.

7. If a traveler is in latitude $-4°$ and travels north 7°, what does his latitude become? What does it become if instead he travels south 7°?

8. If a man's property is $-\$7000$ and he saves \$2000 a year for 8 years, what does his property become?

9. If a vessel, at latitude 3°, sails south 345 miles, what does her latitude become if 60 miles equal 1°?

10. If a man bought a horse for \$150 and sold it for \$200, what was his gain? What would his gain have been if he had sold it for \$125? For \$100?

11. What is meant by saving $-$ \$10? By a distance $-$ 10 miles north?

12. What is the absolute value of -4 miles? Of $+4$ miles? -5 inches? $-3°$? $-$ \$4200?

13. Make up an example for yourself showing the meaning of absolute value.

(The following problems are variations of **Type II**, or are of **Type III**, viz.: $\mathbf{x + ax + b = c}$.)

14. Walter and his brother together had 90 marbles, and his brother had 10 less than Walter. How many marbles had each boy?

Let x = no. of marbles Walter had
Then $x - 10$ = no. of marbles his brother had
$x + x - 10 = 90$
$2x - 10 = 90$
Adding 10 to each of these equals (**Art. 15, 3**)
$2x = 100$
x = 50, *no. of marbles Walter had*
$x - 10$ = 40, *no. of marbles his brother had*

15. A basket ball team has played 27 games and has lost 3 less than it has won. How many games has it won?

16. In a certain election 12,420 votes were cast, and the defeated candidate had 210 less votes than the winning candidate. How many votes had each candidate?

17. Make up and work a similar example for yourself.

18. Walter and his brother together have 83 marbles. If his brother has 7 less than twice the number Walter has, how many has each boy?

19. One number exceeds 4 times another number by 8, and the sum of the numbers is 100. Find the numbers.

20. One number exceeds 3 times another number by .12, and the sum of the numbers is 4.4. Find the numbers.

21. One fraction exceeds 5 times another fraction by $\frac{1}{9}$, and the sum of the fractions is $\frac{13}{9}$. Find the fractions.

22. The distance from New York to Chicago is 912 miles. If this is 24 miles less than 4 times the distance from New York to Boston, what is the latter distance?

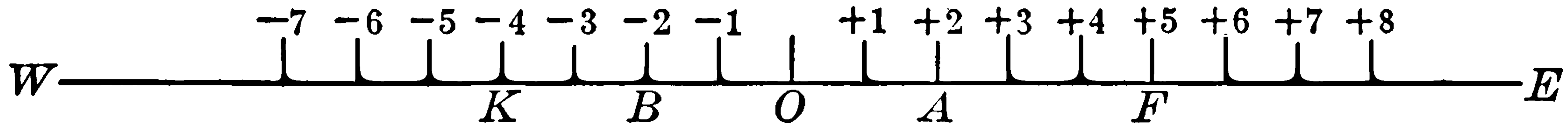
23. The Eiffel Tower is 984 ft. high. If this is 126 ft. less than twice the height of the Washington Monument, what is the height of the Washington Monument?

24. How many of the examples in this Exercise can you work at sight?

25. Which of Exs. 14–23 are of type $x + x - a = b$, and which are of type $x + ax \pm b = c$?

26. Make up and work an example similar to Ex. 19. To Ex. 23.

26. Double Use of + and − Signs. The signs + and − are employed for two purposes (see Arts. 7 and 23): first, to indicate the operations of addition and subtraction; and second, to express positive and negative quantity. We are able to make this double use of these signs because, in each use, the signs are governed by the same laws.



A person walks from O toward E a distance of 5 miles (to F) and then walks back toward W a distance of 3 miles (to A). If the distance to the right of O is regarded as positive, and therefore the distance to the left of O is negative, the distance from the starting point to the destination may be expressed as the sum of a positive quantity and a negative quantity; that is,

$$\text{(positive distance } OF) + \text{(negative distance } FA),$$

$$\text{or, } +5 + (-3) = 5 - 3 = 2.$$

The position arrived at may be determined in another way — viz. by deducting 3 miles from 5 miles. We obtain

$$5 - (+3) = 5 - 3 = 2.$$

From this example we see that *adding negative quantity is the same in effect as subtracting positive quantity.*

Therefore, in the expression $5 - 3$, the minus sign may be considered either a sign of the *quality* of 3, or as a sign of *operation* to be performed on 3. Hence, we are able to use the signs $+$ and $-$ to cover two meanings.

27. Laws for the Use of + and − Signs. Whichever of the two meanings of $+$ and $-$ named in Art. 26 is assigned, we see that $+(-3) = -3$; also, $-(+3) = -3$.

The signs + and − applied in succession to a quantity are equivalent to the single sign −.

Or in symbols,

$$+(-\mathbf{a}) = -\mathbf{a}; \text{ and } -(+\mathbf{a}) = -\mathbf{a}.$$

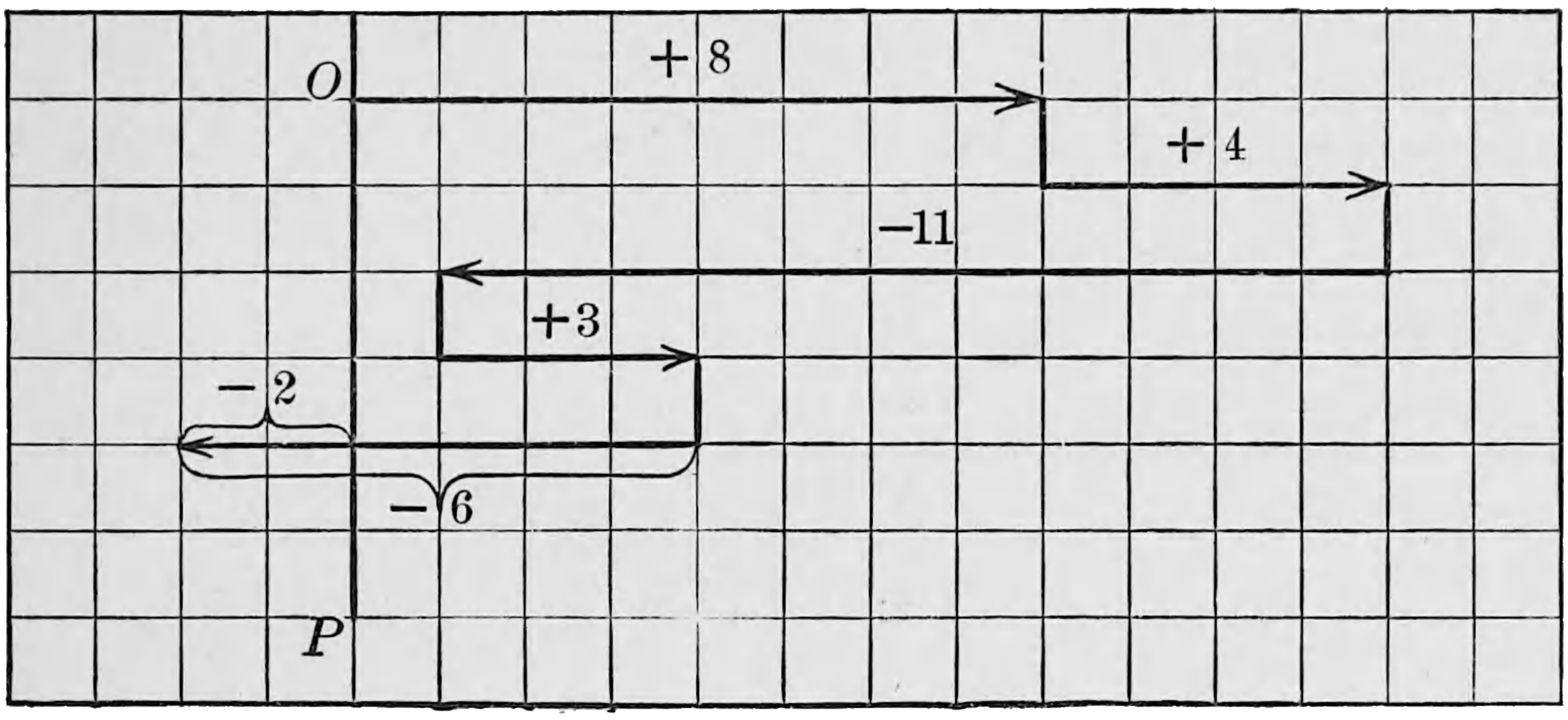
Ex. Find the value of $8 + 4 - 11 + 3 - 6$. On squared paper show the meaning of the numbers involved.

$$8 + 4 - 11 + 3 - 6 = 15 - 17 = -2 \textit{ Ans.}$$

Taking the distances to the right of OP as positive, we have the diagram on p. 33 showing the meaning of the numbers involved.

Note that the above process holds true whether a number preceded by a minus sign is regarded as the subtraction of a positive number or the addition of a negative number.

If in the illustration on p. 31 a person walks in the negative direction from O (i. e. toward W) a distance of 4 miles to K, and then reverses his direction and goes 2 miles, he will be at B. Or stated in another way, diminishing the distance traveled west by 2 miles, brings him to the same place as walking the full direction west and then walking 2 miles east.



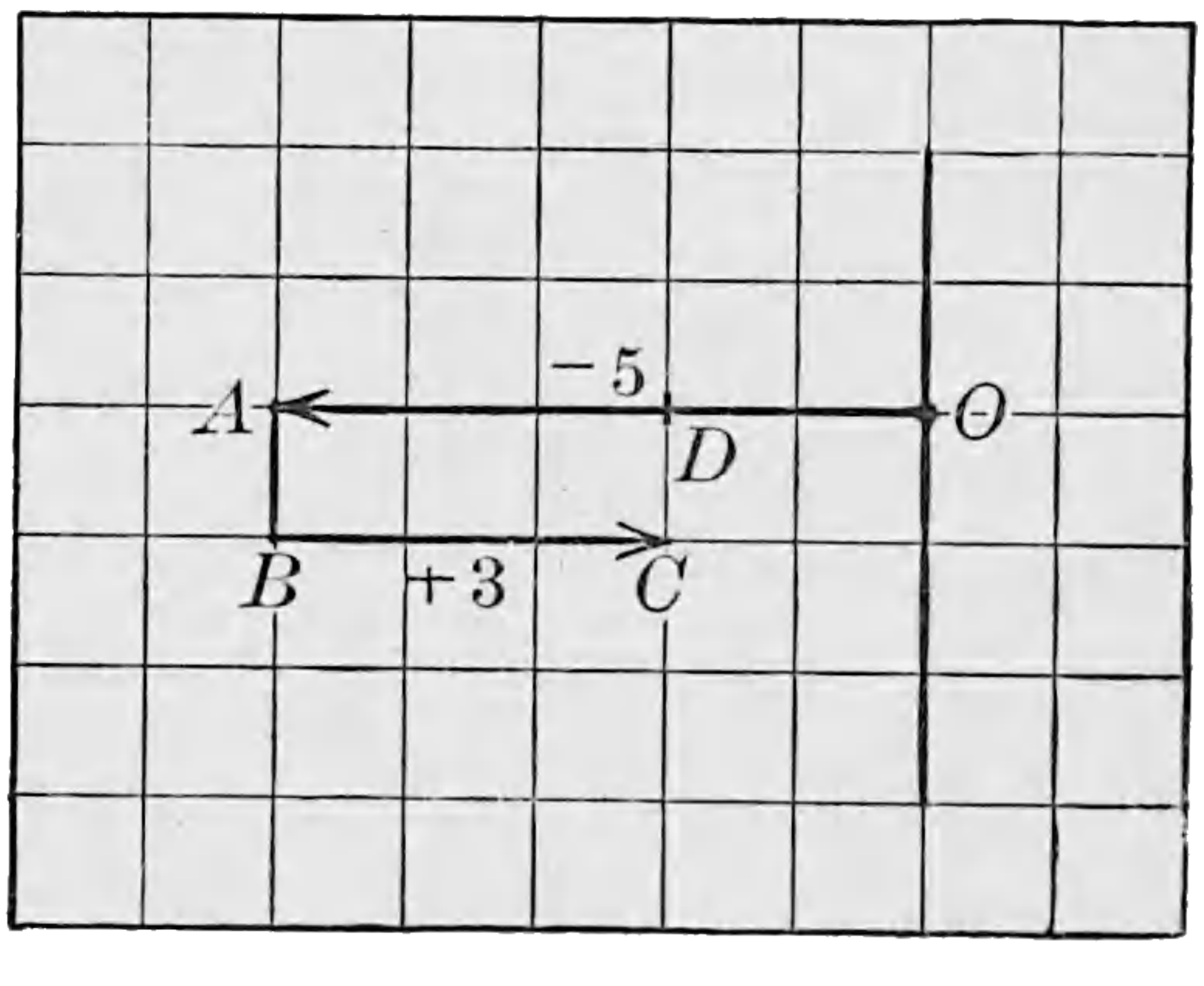
It may be well to study another illustration of this principle. If a man owes two notes of \$500 and \$100 respectively, removing the note for \$100 is the same in effect as annexing \$100 in money to the debts as they are. That is,

$$-\$500-\$100-(-\$100)=-\$500-\$100+\$100=-\$500$$

Hence:

The sign − applied twice to a given positive quantity gives a + result.

Or in symbols, $-(-\mathbf{a})=+\mathbf{a}$.



These laws enable us to *use negative quantity with as great freedom as we use positive quantity,* and hence are an important source of power, as will become more evident later.

Ex. On squared paper show the meaning of $-5-(-3)$. Also of $-5+3$. Hence, show that $-5-(-3)=-5+3$.

On the lower diagram on p. 33 $-5-(-3)$ means $OA - DA$, or OD. Also $-5+3$ means $OA + BC$, or OD.

Hence, $-5-(-3)$ and $-5+3$ give the same result; or we may say $-5-(-3) = -5+3$.

28. The Algebraic Sum of two or more algebraic numbers is the result of combining the given algebraic numbers into a single number.

Thus, the algebraic sum of 4 and -7 is -3.

EXERCISE 7

Find the value of each of the following and verify the result on squared paper:

1. $5-2$.
2. $6-8$.
3. $5-5$.
4. $-4+2$.
5. $-4-2$.
6. $0-4$.
7. $8-6-4$.
8. $7-5+4$.
9. $3+1-5$.
10. $-4-(-3)$.
11. $5-(-8)$.
12. $-7+(-2)$.
13. $0-(-5)$.
14. $0+(-5)$.
15. $-4-(-1.5)$.
16. $4+5-12+3-5$.
17. $-3+8-6-2+2-1$.

18. At 6 A. M. a thermometer read 57°. It then made successive changes as follows: $+7°$, $-2°$, $+5°$, $-3°$, $-2°$. What was the final reading?

19. In a certain football game, taking a distance toward the north goal as positive, during the first seven plays the ball started at the middle of the field and shifted its position in yards as follows: $+50-10-15-5+10-5-20$. Find the final position of the ball with reference to the middle of the field. On squared paper show the changes in the position of the ball, letting 5 yd. equal one space on the paper.

20. State in the language of debts and credits the meaning of

$$-\$700-\$200-(-\$200)=-\$700-\$200+\$200$$

Sug. If a man has debts of $700 and $200, the removal of the $200 debt is the same as leaving his debts unchanged and adding $200 to his possessions. He becomes worth – $700 in either case.

21. State in the language of distance traveled east and west the meaning of

$$-10\text{ mi.}-2\text{ mi.}-(-2\text{ mi.})=-10\text{ mi.}-2\text{ mi.}+2\text{ mi.}$$

(The following are miscellaneous problems of **Type II** and **Type III.**)

22. A man and a boy together catch 320 fish, and the man receives three times as many fish as the boy. How many fish does each have?

23. A man has $3220 in two banks and the amount in one bank exceeds that in the other by $540. How much has he in each bank?

24. Two girls make $24.60 by sewing, and the younger girl receives only one half as much as the older. How much does each receive?

25. Separate $12.68 into two parts one of which shall be smaller than the other by $5.

26. A given piece of bronze weighs 4600 lb. It contains twice as much tin as zinc, and $8\frac{1}{2}$ times as much copper as zinc. How many pounds of each metal does the bronze contain?

27. The distance from the mouth of the Mississippi River to the source of the Missouri River is 4500 miles. The distance between the mouth of the Mississippi and the mouth of the Missouri is 1700 miles less than the length of the Missouri. What is the length of the Missouri?

28. A farmer obtained 2720 pounds of cream in one month by the use of a separator. This is $\frac{1}{5}$ more than he would have obtained if his milk had been skimmed by hand. How much would he have obtained by the latter process?

29. The cost of a macadam road was $24,000. The county paid twice as much as the state, and the township three times as much as the state. How much did each pay?

30. Three partners divided $14,000, the second partner receiving $2000 more than the first, and the third partner receiving twice as much as the first. How much did each receive?

31. Mt. Washington is 6290 ft. high, This is 170 ft. more than 10 times the height of the Singer Building (N. Y.). How high is the latter?

32. Make up and work an example similar to Ex. 18. Ex. 21. Ex. 24. Ex. 29.

33. How many of the examples in this Exercise can you work at sight?

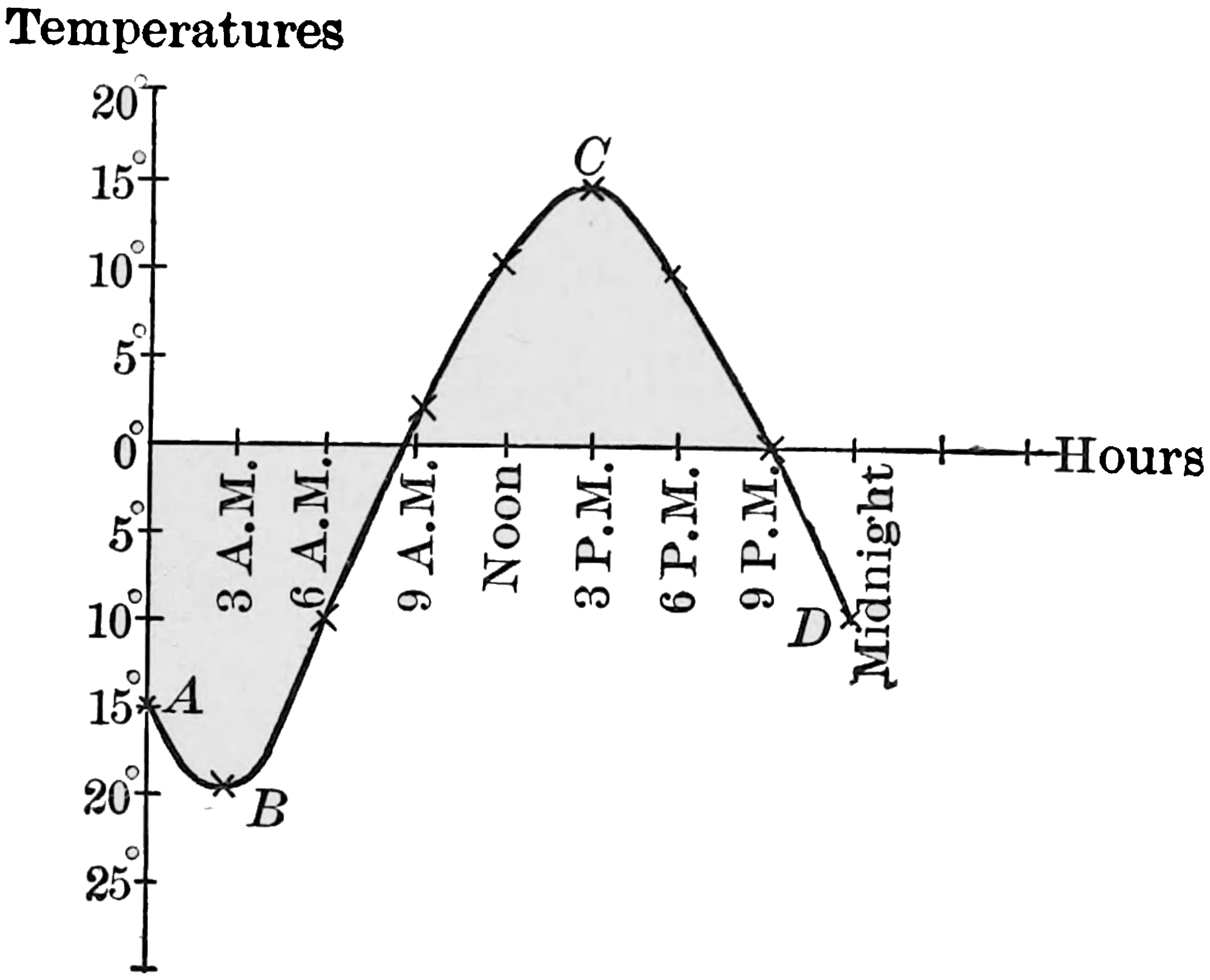
34. Which of Exs. 22–33 of this Exercise are of Type I? Of Type II? Type III?

29. Graphs. A set of numerical facts may often be combined as a geometrical picture called a *graph*. The meaning and use of negative numbers are often well illustrated on a graph.

Ex. On a given day the following were the temperatures at a given place:

Midnight	– 15°	9 A. M.	2°	6 P. M.	10°
3 A. M.	– 20°	Noon	10°	9 P. M.	0°
6 A. M.	– 10°	3 P. M.	15°	Midnight	– 10°

Graph these facts.



We draw a horizontal line and on it mark off spaces to represent hours, as in the diagram. Perpendicular to this we draw a line and on it mark off spaces to represent temperatures. Above or below each point which represents an hour, a point is located which represents the temperature at that hour. Through the points thus located a continuous line *ABCD* is drawn. This is the required graph.

EXERCISE 8

Graph each of the following sets of temperatures:

	Midnight	3 A. M.	6 A. M.	9 A. M.	12 M.	3 P. M.	6 P. M.	9 P. M.
1.	–20°	–30°	–20°	–10°	0°	10°	10°	0°
2.	–10°	–20°	–10°	0°	10°	20°	10°	0°
3.	–10°	–15°	– 5°	10°	15°	25°	15°	–5°
4.	0°	–10°	– 5°	15°	25°	30°	15°	5°

5. Make up and work a similar example for yourself.

Graph each of the following sets of temperatures:

		Jan. 1	Feb. 1	Mar. 1	Apr. 1.	May 1	June 1
6.	New York	31°	31°	35°	42°	54°	64°
7.	New York	–1° C.	–1° C.	1° C.	6° C.	12° C.	18° C.
8.	London	37°	38°	40°	45°	50°	57°

July 1	Aug. 1	Sept. 1	Oct. 1	Nov. 1	Dec. 1
71° 22° C. 62°	73° 23° C. 62°	69° 21° C. 59°	61° 16° C. 54°	49° 9° C. 46°	39° 4° C. 41°

9. Convert the temperatures given for London in Ex. 8 to temperatures on the Centigrade scale and graph them (see Ex. 9, p. 26).

10. Collect and graph sets of numerical facts similar to those given in the preceding examples.

CHAPTER III

ADDITION AND SUBTRACTION; THE EQUATION

ADDITION

30. The Utility of Addition in Algebra.

Ex. Find the value of $3ab^2 + 5ab^2 + 2ab^2$, when $a = 2$ and $b = 3$.

PROCESS WITHOUT ALGEBRAIC ADDITION

If we substitute directly in the given expression, we obtain

$$\begin{aligned} 3ab^2 + 5ab^2 + 2ab^2 &= 3 \times 2 \times 3^2 + 5 \times 2 \times 3^2 + 2 \times 2 \times 3^2 \\ &= 54 + 90 + 36 \\ &= 180 \textit{ Ans.} \end{aligned}$$

PROCESS AIDED BY ALGEBRAIC ADDITION

$$\begin{aligned} 3ab^2 + 5ab^2 + 2ab^2 &= 10ab^2 \\ &= 10 \times 2 \times 3^2 \\ &= 180 \textit{ Ans.} \end{aligned}$$

In solving the above example, algebraic addition enables us to save more than half the work. Algebraic addition has other uses which will appear later.

Why do we now make definitions and rules?

31. Addition, in algebra, is the combination of several algebraic expressions into a single equivalent expression.

Addition is sometimes described as *collecting terms* in an expression.

32. Similar Terms (or like terms) are terms which contain the same literal factors and the same radical signs over the same factors.

Thus, $7ab^2$ and $-5ab^2$ are similar terms. Also $5a\sqrt[3]{2}$ and $-6a\sqrt[3]{2}$ are similar terms.

Dissimilar terms (or unlike terms) are terms which are unlike either in their literal factors or in the radical sign over the same factor.

Thus, $5a^2b$ and $5ab^2$ are dissimilar terms. Also $3\sqrt[3]{5}$ and $3\sqrt{5}$ are dissimilar terms.

The addition of dissimilar terms can only be indicated.

Thus, b added to a gives $a + b$; also $a^3 - 3a^2b$ added to $3a^2 - b^3$ gives $a^3 - 3a^2b + 3a^2 - b^3$.

33. Method for Addition. The most convenient general method for addition is shown in the following examples:

Ex. 1. Add $4x^2 + 3x + 2$, $3x^2 - 4x - 3$, $-2x^2 - x - 5$.

Arranging similar terms in the same column, and adding each column separately, we obtain

$$\begin{array}{rrcrcr}
 & & & \text{CHECK} & & \\
 & 4x^2 + 3x + 2 & = & 4 + 3 + 2 & = & 9 \\
 & 3x^2 - 4x - 3 & = & 3 - 4 - 3 & = & -4 \\
 & -2x^2 - x - 5 & = & -2 - 1 - 5 & = & -8 \\
\hline
\textit{Sum} & 5x^2 - 2x - 6 & = & 5 - 2 - 6 & = & -3
\end{array}$$

To check the accuracy of the work, we let x = any convenient number, as 1; find the numerical value of each row; and compare the sum of these results with the numerical value of the algebraic expression obtained as the sum.

Ex. 2. Add $2a^3 - 5a^2b + 4ab^2 + a^2b^3$, $4a^2b + 2a^3 - ab^4 - 3ab^2$, $a^2b - a^3 + 2ab^2$.

Proceeding as in Ex. 1,

$$\begin{array}{rlcrcr}
 & & & \text{CHECK} & & \\
 & 2a^3 - 5a^2b + 4ab^2 + a^2b^3 & = & 2 - 5 + 4 + 1 & = & 2 \\
 & 2a^3 + 4a^2b - 3ab^2 \qquad - ab^4 & = & 2 + 4 - 3 - 1 & = & 2 \\
 & -a^3 + a^2b + 2ab^2 & = & -1 + 1 + 2 & = & 2 \\
\hline
\textit{Sum} & 3a^3 \qquad + 3ab^2 + a^2b^3 - ab^4 & = & 3 + 0 + 3 + 1 - 1 & = & 6
\end{array}$$

In the second column the algebraic sum of the coefficients is $-5 + 4 + 1$, which $= 0$; and as zero times a number is zero, the sum of the second column is zero, which need not be set down in the result.

The work is checked by letting a and b each $= 1$.

Hence, the process for addition may be stated as follows:

Arrange the terms to be added in columns, placing similar terms in the same column;

Find the algebraic sum of the numerical coefficients of each column and prefix this result to the literal factors common to the terms in the column.

Sometimes the algebraic sum of the coefficients of each group of similar terms is found without arranging the terms in columns.

EXERCISE 9

Add and check each result:

1.	**2.**	**3.**	**4.**	**5.**
-11	4	$8x$	$-x$	$-7x$
6	-10	$-6x$	$-3x$	$12x$

6.	**7.**	**8.**	**9.**	**10.**
$2a$	$-x^2$	$7xy$	a^2b	$7x^2y^2$
$5a$	$3x^2$	$-10xy$	$5a^2b$	$-10x^2y^2$
$-12a$	$5x^2$	$2xy$	$-3a^2b$	x^2y^2

11. $3ax, -2ax, 5ax, ax, -3ax.$

12. $5x^2, 12x^2, -10x^2, x^2, -16x^2, 3x^2, -x^2.$

13. $7a^2b^2, -12a^2b^2, -a^2b^2, -4a^2b^2, 5a^2b^2, 6a^2b^2.$

14.	**15.**	**16.**	**17.**
$3(a+b)$	$-6(x-y)$	$5\sqrt{a+x}$	$4\pi r^2$
$5(a+b)$	$4(x-y)$	$-6\sqrt{a+x}$	$-2\pi r^2$
$-4(a+b)$	$-5(x-y)$	$2\sqrt{a+x}$	$\frac{1}{2}\pi r^2$

18.

$$\begin{array}{r} 3x - 2y \\ -2x + 3y \\ x - y \\ \hline \end{array}$$

19.

$$\begin{array}{r} 5x^2 + 7 \\ x^2 - 10 \\ -7x^2 + 1 \\ \hline \end{array}$$

20.

$$\begin{array}{r} a^2 - ax + 4x^2 \\ 3a^2 + 2ax - 5x^2 \\ -a^2 - ax - x^2 \\ \hline \end{array}$$

21. $a - 2b, 3a + 4b, a + 5b, -5a - b, a - 5b.$

22. $3x^2 + y^2, 2x^2 - 7y^2, -4x^2 - 5y^2, x^2 + 3y^2, -3y^2.$

23. $3ax^3 - 5by^3, 2ax^3 + 4by^3, 2by^3 - 4ax^3, by^3 - ax^3.$

Reduce each of the following to its simplest form:

24. $x^2 - xy + 3y^2 + 2x^2 + 2xy - 2y^2 + x^2 + y^2 + 3x^2 - xy.$

25. $mn - 3n^2 + m^2 + m^2 + 2n^2 - 3mn + m^2 - n^2 + mn - 2m^2.$

26. $x^2 + y^2 - 2z^2 + 3x^2 - y^2 + 2z^2 + z^2 - 2x^2 + x^2 - z^2.$

27. $2x^2 - xy + 3xy - 5y^2 + 3y^2 - 3x^2 + x^2 + 2y^2 - 2xy.$

28. $7x + y + 5z - 10xy + 2y - 3z + 13xy - 4xz + 5z - 6x - 4xz + 2xy - 3y + 9z + 7x - xz + 21xz - 16z + x - 5xy.$

29. $x^3 + 3x^2y + 3xy^2 + y^3 + x^3 - 3x^2y + 3xy^2 - y^3 + 2x^2y - 2xy^2 + y^3 + x^3 - y^3 + x^2y - 4x^3 - xy^2 - y^3 + y^3 + x^3 - x^2y + xy^2.$

Collect similar terms in the following and check each result:

30. $2x - 3y - 5x + 4z + 4y + z - 2y - x - 3z + 2x - 3y.$

31. $3xy - 5ax + 3y^2 - 2xy - 3x^2 + 4ax - 2y^2 + 3ax - 2xy.$

32. $x - 3y + 2z + 2y - 2x - z - 3x - 4z - 2x + z + 2x.$

33. $2x - 1 + 5y - 2 + 3x + 2 + 3y - 3 - 2x + 1 - x - 3y.$

34. $3a^2b - 2a^2c + 3a^2 - 5a^2b - a^2 - 3a^2c + a^2b + 6a^2c - 2a^2.$

35. $5x^3 - 3x + 4 - 2x^2 - 6x^3 + 4x - 7 - x^2 + x^3 + 3x^2 - x + 5 + 3x^2 - 6x - x^2 + 4x - 2x^2 + 2x.$

36. $2x^n - 5x^m + 3x^2 - x^n - 7x + 3x^2 - 3 + 2x^m - 5x^2 + 5 + 3x^m.$

37. Reduce $3xxxyy + 8xxxyy - 5xxxyy - 2xxxyy$ to its simplest form. About how much briefer is the form you obtain than the given form?

38. Make up and work an example similar to Ex. 37.

39. Make up and work an example showing the use of algebraic addition (see Ex. of Art. 30, p. 39).

40. State in general language the use of algebraic addition.

(The following are mixed problems of **Types I, II, III.**)

41. Three partners in a retail business made $18,000 in one year. The second partner owned the building and received twice as large a share as the first partner. The third partner supplied most of the capital and received three times as large a share as the first. How much did each receive?

42. Make up and work a similar example for yourself.

43. Find three consecutive numbers whose sum is 36.

44. Find four consecutive numbers whose sum is 106.

45. Make up and work an example concerning five consecutive numbers.

46. The area of the United States and its outlying possessions is 3,742,155 sq. mi. The area of the United States exceeds that of its outlying possessions by 2,309,045 sq. mi. What is the area of the outlying possessions?

47. How many of the examples in this Exercise can you work at sight?

48. Name the type to which each of the above problems belongs (Exs. 41–46).

SUBTRACTION

34. The Utility of Subtraction in Algebra.

Ex. Find the numerical value of $17a^2b^3 - 15a^2b^3$, when $a = 3$ and $b = 2$.

PROCESS WITHOUT ALGEBRAIC SUBTRACTION

$$\begin{aligned} 17a^2b^3 - 15a^2b^3 &= 17 \times 3^2 \times 2^3 - 15 \times 3^2 \times 2^3 \\ &= 17 \times 9 \times 8 - 15 \times 9 \times 8 \\ &= 1224 - 1080 = 144 \textit{ Ans.} \end{aligned}$$

PROCESS AIDED BY ALGEBRAIC SUBTRACTION

$$\begin{aligned} 17a^2b^3 - 15a^2b^3 &= 2a^2b^3 \\ &= 2 \times 3^2 \times 2^3 \\ &= 144 \textit{ Ans.} \end{aligned}$$

In solving the above example, algebraic subtraction enables us to save more than half of the work. Algebraic subtraction has other advantages which will appear later.

Why do we now proceed to make definitions and rules?

35. Subtraction, in algebra, is the process of finding a quantity which, added to a given quantity (the subtrahend), will produce another given quantity (the minuend).

Thus, if we subtract $3ab$ from $10ab$, we obtain $7ab$, for $7ab$ added to $3ab$ (subtrahend) gives $10ab$ (minuend).

36. Signs in Subtraction. From Art. 26 it follows that

Subtracting a positive quantity is the same as adding a negative quantity of the same absolute magnitude; and

Subtracting a negative quantity is the same as adding a positive quantity of the same absolute magnitude.

37. Method for Subtraction. The most convenient general method in subtraction is to

Write the terms of the subtrahend under the terms of the minuend, placing similar terms in the same column;

Change the signs of the terms in the subtrahend mentally, and proceed as in addition.

Ex. 1. From $5x^3 - 2x^2 + x - 3$ subtract $2x^3 - 3x^2 - x + 2$. Check the work by letting $x = 1$.

$$\begin{array}{lrcl}
 & & & \text{CHECK} \\
 & 5x^3 - 2x^2 + x - 3 & = & 5 - 2 + 1 - 3 = 1 \\
 & 2x^3 - 3x^2 - x + 2 & = & 2 - 3 - 1 + 2 = 0 \\
\hline
\textit{Difference} & 3x^3 + x^2 + 2x - 5 & = & 3 + 1 + 2 - 5 = 1
\end{array}$$

The coefficient of x^3 is $5 - 2$, or 3, of x^2 is $-2 + 3$, or 1, etc.

Ex. 2. Subtract $2a^4 - 3a^3b - 6a^2b^2 - 2ab^3 + 2b^4$ from $a^4 + 5a^3b - 6a^2b^2 - 3ab^3$. Check the work by letting $a = 1$ and $b = 1$.

$$\begin{array}{rcrcr}
 & & \text{CHECK} & & \\
a^4 + 5a^3b - 6a^2b^2 - 3ab^3 & = & 1 + 5 - 6 - 3 & = & -3 \\
2a^4 - 3a^3b - 6a^2b^2 - 2ab^3 + 2b^4 & = & 2 - 3 - 6 - 2 + 2 & = & -7 \\
\hline
-a^4 + 8a^3b \qquad\qquad - ab^3 - 2b^4 & = & -1 + 8 + 0 - 1 - 2 & = & 4
\end{array}$$

The coefficient of a^2b^2 is $-6 + 6$, or 0. The coefficient of b^4 is $0 - 2$, or -2.

EXERCISE 10

Subtract and check each result:

1. $\begin{array}{r} 7ab \\ \underline{3ab} \end{array}$ **2.** $\begin{array}{r} 5x \\ \underline{9x} \end{array}$ **3.** $\begin{array}{r} x \\ \underline{2x} \end{array}$ **4.** $\begin{array}{r} 5x \\ \underline{-3x} \end{array}$ **5.** $\begin{array}{r} -3x^2 \\ \underline{-4x^2} \end{array}$ **6.** $\begin{array}{r} -7xy \\ \underline{3xy} \end{array}$

7. $\begin{array}{r} 5(a + b) \\ \underline{3(a + b)} \end{array}$ **8.** $\begin{array}{r} 7(x + y) \\ \underline{-3(x + y)} \end{array}$ **9.** $\begin{array}{r} -2\sqrt{a + x} \\ \underline{-3\sqrt{a + x}} \end{array}$ **10.** $\begin{array}{r} -4\sqrt{b - y} \\ \underline{2\sqrt{b - y}} \end{array}$

11. $\begin{array}{r} 3x^2 - 4x \\ 2x^2 + x \end{array}$ **12.** $\begin{array}{r} 3x - 9 \\ 5x + 1 \end{array}$ **13.** $\begin{array}{r} 2x^3 - 5 \\ -x^3 + 2 \end{array}$ **14.** $\begin{array}{r} 5x^2 + 4x - 3 \\ -x^2 - 3x + 5 \end{array}$

15. From $3a + 2b - 3c - d$ take $2a - 2b + c - 2d$.

16. From $7 - 3x + 2x^2$ take $15 - 4x - 5x^2$.

17. From $x^2 - y^2 - z^2 + 8$ take $2x^2 + y^2 - 2z^2 + 10$.

18. From $5xy - 3xz + 5yz + x^2$ take $4xz - 2xy - x^2$.

19. From $2 - x + x^2 + x^4$ take $3 + x - x^2 - x^3 - 2x^4$.

20. Subtract $10x^2y + 3x^2y^2 - 13xy^2$ from $x^2y - xy^2 + 2x^2y^2$.

21. Subtract $3 - 2ab + 3ac - 4cd$ from $5 - ac + 8cd - 5ad$.

22. Subtract $1 + x - x^2 + x^3 - x^4$ from $2 - x - x^2 - x^3 + x^5$.

23. Subtract $a + 2b - 3c + 4d$ from $m + 2b + d - x + a$.

24. Subtract $3x^4 - 2x^2 + 5x - 7$ from $3x^3 + 2x^2 - x - 7$.

25. Subtract $-x^5 - 2x^4 + x^2 + 5$ from $x^5 - x^3 + x^2 - 2x + 5$.

26. Subtract $3x^m - 3x^n + x - 3$ from $x^m + x^n - x^2 + x - 1$.

27. From the sum of $2x$ and $3y$ subtract their difference.

28. From 0 subtract $-3x$. From 0 subtract $x - y$. From zero subtract $3a^2 - 2ab + b^2$.

29. Reduce $7aaabb + 5aaabb - 3aaabb$ to its simplest form.

30. Make up and work an example similar to Ex. 29.

If $A = x^3 - 3x^2 + 1$, $B = 2x^2 - 5x - 3$, $C = 3x^3 + x^2 + 3x$, find the value of

31. $A + B + C$

32. $B - A + C$

33. $A + B - C$

34. $A - B + C$

(The following problems are variations of **Types I** and **II.**)

35. Find the value of x, if $3x - 2$ in. $= 7$ in.

36. Separate \$24.80 into two parts such that one part is smaller than the other by \$4.60.

37. Separate \$24.80 into two parts such that the smaller part equals $\frac{1}{3}$ of the larger part.

38. Separate \$5000 into three parts such that the second part shall exceed the first by \$300, and the third shall exceed the first by \$800.

39. Separate \$5000 into three parts such that the second part shall exceed the first by \$300, and the third shall exceed the second by \$800.

40. Separate \$6000 into three parts such that the second part equals $\frac{1}{2}$ of the first, and the third part equals $\frac{1}{5}$ of the first.

41. Separate \$6000 into three parts such that the second part is double the first, and the third part is double the second.

42. Make up and work an example similar to Ex. 36. To Ex. 40.

43. Name the type of which each of the above problems (Exs. 36–43) is a variation.

44. How many of the examples in this Exercise can you work at sight?

45. How many of the examples in Exercise 1 can you now work at sight?

USE OF THE PARENTHESIS

38. Utility of the Parenthesis. The parenthesis is useful in indicating an addition or a subtraction in a brief way.

Thus, $2a + 3b - 5c - (3a - 2b + 3c)$ indicates that $3a - 2b + 3c$ is to be subtracted from $2a + 3b - 5c$.

The parenthesis will also be found useful in indicating multiplication and division in a brief manner, and other uses of the parenthesis will become evident as we proceed.

39. Removal of a Parenthesis. From the processes of addition and subtraction it follows that

When a parenthesis preceded by a $+$ *sign is removed, the signs of the terms inclosed by the parenthesis remain unchanged.*

But

When a parenthesis preceded by a minus sign is removed, the signs of the terms inclosed by the parenthesis are changed, the $+$ *signs to* $-$, *and the* $-$ *signs to* $+$.

Ex. Simplify $2a + 3b - 5c - (3a - 2b + 3c)$.

$$\begin{aligned} 2a + 3b - 5c - (3a - 2b + 3c) &= 2a + 3b - 5c - 3a + 2b - 3c \\ &= -a + 5b - 8c \ \textit{Ans.} \end{aligned}$$

Let the pupil check the work by letting $a = 1$, $b = 1$, $c = 1$.

40. Parenthesis within Parenthesis. Using the *parenthesis* as a general name for all the signs of aggregation, it is evident that several parentheses may occur one within another in the same algebraic expression. The best general method of removing several parentheses occurring thus is as follows:

Remove the parentheses one at a time, beginning with the innermost;

Collect the terms of the result.

It is also possible to remove the parentheses in reverse order, that is, by removing the outside parenthesis first, etc. Working an example in this way often forms a convenient check on the first process.

Ex. Simplify $5x - y - [4x - 6y + \{-3x + y + 2z - (2x - z)\}]$.

$$\begin{aligned} &5x - y - [4x - 6y + \{-3x + y + 2z - (2x - z)\}] \\ = {}&5x - y - [4x - 6y + \{-3x + y + 2z - 2x + z\}] \\ = {}&5x - y - [4x - 6y - 3x + y + 2z - 2x + z] \\ = {}&5x - y - 4x + 6y + 3x - y - 2z + 2x - z \\ = {}&6x + 4y - 3z \ \textit{Ans.} \end{aligned}$$

The work may be checked by removing the parentheses in reverse order, or by the method of substitution as follows:

Letting $x = 1$, $y = 2$, $z = 3$, we have

$$\begin{aligned} &5x - y - [4x - 6y + \{-3x + y + 2z - (2x - z)\}] \\ &= 5 - 2 - [4 - 12 \{-3 + 2 + 6 - (2 - 3)\}] \\ &= 3 - [-8 + \{5 - (-1)\}] = 3 - [-8 + \{5 + 1\}] \\ &= 3 - (-8 + 6) = 3 - (-2) = 3 + 2 = 5 \end{aligned}$$

Also $6x + 4y - 3z = 6 + 8 - 9 = 5$

EXERCISE 11

Remove parentheses and collect similar terms. Check each result either by substitution of numerical values, or by reversing the order in which the parentheses are removed.

1. $3a + (2a - b)$.

2. $2x - (x - 1)$.

3. $x + (1 - 2x)$.

4. $3x - (1 + 3x)$.

5. $x - (-x - 1)$.

6. $x + 2y - (2x - y)$.

7. $x - [2x + (x - 1)]$.

8. $5x + (1 - [2 - 4x])$.

9. $2 - \{1 - (3 - a) - a\}$.

10. $2x - [-x - (x - 1)]$.

11. $2y + \{-x - (2y - x)\}$.

12. $a - \{-a - (-a - 1)\}$.

13. $[x^2 - (x^2y - z^2) - z^2] + (x^2y - x^2)$.

14. $1 - \{1 - [1 + (1 - x) - 1] - 1\} - x$.

15. $x - [-\{-(-x - 1) - x\} \underline{- 1]} - 1$.

16. $1 - \{2 + [-3 - (-\underline{4 - 5 - 6)} - 7]\}$.

17. $a - \{a + [b - (a + b + c - a + b + d) - c]\}$.

18. $x - \{2x^2 + (3x^3 - 3x - [x + x^2]) + [2x - (x^2 + x^3)]\}$.

19. $x^4 - [4x^3 - [3x^2 - (2x + 2)] + 3x] - [x^4 + (3x^3 + 2x^2 - 3x - 1)]$.

20. $-[-2x - \{-(-2x - 1) - 2x\} - 1] - 2x$.

21. $x - [x + (x - y) - \{x + (y - x) - 2y\} + y] - y + x$.

22. $25x - [12 + \{3x - 7 - (-12x - \overline{5 + 15x}) - (3 + 2x)\}] + 7 - (3x + 5) + (2x - 3) + x + 8$.

23. In $3x - (5a - 2b + c)$, what is the sign of $5a$ as the example stands?

24. In Ex. 12, Exercise 10, indicate the subtraction by use of a parenthesis. Do the same in Ex. 13.

Remove the parentheses and find the value of x in each of the following:

25. $x + (x + 2) = 7$.

26. $3x - (x + 2) = 8$.

27. $5x - (2x - 3) = 12$.

28. $4x - (x - \frac{2}{3}) = 2\frac{1}{4}$.

29. Make up and work an example similar to Ex. 16. To Ex. 26.

30. How many of the examples in this Exercise can you work at sight?

31. How many of the examples in Exercise 3 (p. 19) can you work at sight?

41. Insertion of a Parenthesis. It is clear that the process of removing a parenthesis may be reversed; that is, that terms may be inclosed in a parenthesis.

Inverting the statements of Art. 39, we have

Terms may be inclosed in a parenthesis preceded by the plus sign, provided the signs of the terms remain unchanged;

Terms may be inclosed in a parenthesis preceded by the minus sign, provided the signs of the terms are changed.

Ex. $a - b + c + d - e = a - b + (c + d - e)$,

or, $= a - b - (-c - d + e)$ *Ans.*

EXERCISE 12

In each of the following insert a parenthesis inclosing the last three terms, each parenthesis to be preceded by a minus sign. Check the work either by removing the parenthesis in the answer, or by numerical substitution.

1. $x^3 - 3x^2 + 3x - 1.$
2. $a - b + c + d.$
3. $1 + 2a - a^2 - 1.$
4. $1 - a^2 - 2ab - b^2.$
5. $x^4 + 4x - x^2 - 4.$
6. $a^2b^2 - 2cd - c^2 - d^2.$
7. $4x^4 - 9x^2 + 12xy - 4y^2.$
8. $x^4 - 4x^3 + 4x^2 + 4x - 4 - x^2.$

It is often useful to collect the coefficients of a letter into a single coefficient.

Ex. Collect the coefficients of x, y, and z in the expression,

$$3x - 4y + 5z - ax - by - cz - bx + ay + az.$$

The complete coefficient of x is $(3 - a - b)$; of y, $(-4 - b + a)$ or $-(4 + b - a)$; of z, $(5 - c + a)$.

Hence, the expression may be written,

$$(3 - a - b)x - (4 + b - a)y + (5 - c + a)z \quad \textit{Ans.}$$

In like manner collect the coefficients of x, y, and z:

9. $mx - ny + 3z + 2x + nz - 4y.$
10. $x - y - 2z - ax + by - az - bx - ay + cz.$
11. $-7x + 12y - 10z - 2ax + 3bz - cy + 2bx - 6dy.$
12. $5y - 3acx - 5cdz - 4abx - 3cdy + 2cx - 4z - 5ax.$

Collect the coefficients of x^3, x^2, and x:

13. $3x^3 + x - 2x^2 - ax^3 - 5 + ax^2 - 2ax - cx^3 - cx^2 - cx.$
14. $-x^2 - x - ax^3 + x^3 - ax + bx^2 - ax^2 - 3bx - 2bx^3 + 3a.$
15. $a^2x^2 - ax - a - b^3x^3 - 2b^2x^2 + 3bx - a^3x^3 - cx^2 + 3cx - c.$

EQUATIONS AND TRANSPOSITION

42. An Equation is a statement of the equality of two algebraic expressions.

An equation, therefore, consists of the sign of equality and an algebraic expression on each side of it; as $3x - 1 = 2x + 5$.

The **solution of an equation** is the process of finding the value of the unknown number (as of x) in the given equation.

43. Members of an Equation. The algebraic expression to the left of the sign of equality is called the *first member* of the equation; the expression to the right of the sign of equality is called the *second member*.

Thus, in the equation $3x - 1 = 2x + 3$,
the first member is $3x - 1$; the second member is $2x + 3$.

The members of an equation are sometimes called *sides* of the equation.

The members of an equation are similar to the pans of a set of weighing scales which must be kept balanced. (See Art. 15, p. 18.)

44. Utility of Equations. An equation expresses the relation of at least one unknown quantity to certain known quantities. By means of an equation, we are often able to determine the value of the unknown quantity.

See the problems solved in Exercises 1, 2, etc., by the aid of equations.

45. The Transposition of a Term is moving the term from one member of an equation to the other member. We shall see that when a term is transposed, the sign of the term must be changed.

Ex. 1. Find the value of x in $x-5 = 7$.

PROCESS WITHOUT TRANSPOSITION

We have given $x - 5 = 7$

$$\begin{aligned} x - 5 &= 7 \\ 5 &= 5 \\ \hline x &= 7 + 5 \\ \text{or } x &= 12 \end{aligned}$$

Adding 5 to each of the equals, $x = 7 + 5$ (Art. 15, 3)

or $x = 12$ *Ans.*

PROCESS WITH TRANSPOSITION

We have given $x - 5 = 7$

Transferring 5 to the right-hand member of the equation and changing its sign, $x = 7 + 5$

$x = 12$ *Ans.*

Hence transposition is a short way of adding equal numbers to the two members of an equation. The labor saved by means of transposition is more evident when several terms are to be transposed at the same time.

For the present, however, in order to fix firmly in mind the nature of the process, we shall not transpose terms, but shall add equals to the members of an equation when we wish to transfer terms from one member to the other.

Ex. 2. Solve $5x - (x + 2) = 3x - (2x - 7)$.

Removing parentheses, $5x - x - 2 = 3x - 2x + 7$

Adding $-3x + 2x + 2$ to each member, $-3x + 2x + 2 = -3x + 2x + 2$

$$\begin{aligned} 5x - x - 3x + 2x &= 2 + 7 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Ans.

46. Checking the Solution of an Equation. The result obtained by solving an equation may be checked by substituting in each member of the original equation the value of x obtained by the solution. If the two members reduce to the same number, the value found for x is correct.

Thus, in Ex. 2, putting 3 in the place of x,

The left member, $5x - (x + 2) = 15 - (3 + 2) = 15 - 5 = 10$

Also the right member, $3x - (2x - 7) = 9 - (6 - 7) = 9 + 1 = 10$

EXERCISE 13

Solve the following equations without transposition of terms. Verify each result obtained.

1. $x + 2x - 3 = 6.$ ✓
2. $3x = x + 10.$
3. $5x - 1 = 14.$
4. $4x - 3 = 12 - x.$
5. $5x - 1 = 3x + 7.$
6. $3x - 2 = 2x + .74.$
7. $5x - (2x - 3) = 6.$
8. $7x - (5x + 4) = -2.$
9. $9x = 10 - (x + 5).$
10. $8x + (3x - 4) = 25.$
11. $10x - (x - 5) = 4 - (x + 2).$
12. $10 - (3x - 5) = 8 - (7x + 2).$
13. Solve Exs. 1–12 by aid of transposition of terms.

Solve the following problems and check each result:

14. If 5 times x equals 9 diminished by twice x, find x.

15. If $\frac{2}{3}$ of x equals 12 less $\frac{1}{3}x$, find x.

16. If 12 is added to a given number, the result equals three times the given number. Find the number.

17. One number exceeds another by 5 and the sum of the numbers is 12. Find the numbers.

18. The difference of two numbers is 5 and the sum of the numbers is 13. Find the numbers.

19. Separate 12 into two parts such that one part exceeds the other by 5.

20. One number exceeds another by 1.4 and the sum of the numbers is 16.4. Find the numbers.

21. The difference of two numbers is 1.4 and the sum of the numbers is 16.4. Find the numbers.

22. Separate 16.4 into two parts such that one part exceeds the other by 1.4.

23. Make up and work three examples similar to Exs. 14–16. Also to Exs. 17–20.

24. Find three consecutive odd numbers whose sum is 45. Also five consecutive odd numbers whose sum is 45.

25. Find three consecutive even numbers whose sum is 60. Also five consecutive even numbers whose sum is 60.

26. Make up and work an example similar to Ex. 24.

27. Make up and work an example similar to Ex. 25.

28. To what type does each of the above problems belong, or of what type is each a variation?

EXERCISE 14

REVIEW

1. Find the value of $a + 3(b - x)$, when $a = 5$, $b = 2$, and $x = 1$.

2. Find the value of $3x - (x - 2)^2 + 2(x + 1)(4 - x) - \sqrt{5x + 1}$, when $x = 3$.

3. If $s = vt + \frac{1}{2}gt^2$, find the value of s when $v = 10$, $g = 32.16$, and $t = 4$.

4. If $x = 3$, find the value of $4x^2$. Also of $(4x)^2$.

Simplify:

5. $2^4 - 5x^3 - 3x^2 + 2x - 5 + 2x^3 - 3x^4 - 2x + 2x^2 - 2x + 2x^2 - 6 + 3x^2 + x^4 - 3x^3 + 7 - x + 2 + 3x^3 + 2x^4 - 4x - 2x^2$.

6. $3\sqrt{2} - 5\sqrt{3} + 8 + 5\sqrt{3} - 2\sqrt{2} - 7 + 3\sqrt{3} - 4\sqrt{2} - 2$.

Subtract:

7. $3x^3 - 2x^2 + 5x - 3$ from $8x^3 - x^2 - 1$.

8. $5x^3 - 3x^2y + y^3$ from $3x^3 + 7xy^2 - y^3$.

Simplify and collect:

9. $3x - \{-2x + [-4x - (x - 2) - x] - x\} - 1$

10. $9x - \{-8x - [7x + (-6x + 1) - 5x] - 4x\} - (3x + 1) - 2x.$

Bracket coefficients of like powers of x:

11. $x^5 - x^3 + 2 - 3x^4 - ax^3 + ax^5 - cx^4 - 2ax^2 + 3cx^3 - 2cx^5 - 5x^2.$

12. $1 - x - x^2 - x^3 + 2a - 2ax + 2ax^2 - 2ax^3 - 3bx + 3bx^2 + 3bx^3 + cx.$

Solve and give the reason for each step:

13. $3x - 5 = x + 7.$

14. $5x - (x - 4) = 16.$

15. $4x + (x - 1) = 3x - (x + 2).$

16. $3 - (x - 2) = 7 - 5x.$

17. Subtract $5x^2 - 3ax - 2a^2$ from $-3x^2 + 2ax^2 - a^4$.

18. Find the value of $5x^2 - 3(a - 2x) + 5a^2$, when $a = 4$ and $x = 1$.

19. Add $5x^2 - 3ax + 4a^2$, $5ax - 3x^2 + a^2$, and $3ax - x^2 - 2ax$.

20. Simplify $x^2 - [5ax + (a^2 - 2x^2 - ax) - 3x^2] - 5a^2$. Test the accuracy of your work by letting $a = 1$ and $x = 2$.

21. Solve $5 - x = 4 - (7 + 3x)$.

22. The land surface of the world is 51,240,000 square miles. If the land area of the rest of the world is seven times that of North America, find the area of North America.

23. Add $\frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{8}$, $\frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{2}$, and $\frac{3}{4}x^2 - \frac{1}{2}x + \frac{3}{8}$.

24. Subtract $\frac{1}{8}x^2 - \frac{1}{2}x + \frac{3}{8}$ from $\frac{1}{2}x^2 - \frac{3}{4}x - \frac{3}{8}$.

25. Add $.5a^2 - .15a + 2.5$, $1.2a^2 + .3a - 1.5$, and $-.75a^2 + .3a - .7$.

26. Subtract $.27a^2 - .12a - 2.3$ from $1.5a^2 + 2a - 1.7$.

27. Add $2(x + y) - 3(x + z) + 2(y + z)$, $4(x + z) - 3(x + y) - 5(y + z)$, and $4(x + y) - (x + z) + 4(y + z)$.

28. From the sum of $a^2 - 7ab + 3b^2$ and $2a^2 - 6b^2 + 7a^2b^2$, take the sum of $4a^2b^2 - 3a^3 + 2a^2 - b^2$ and $3ab - 2b^2 + a^2$.

29. What must be added to $x^2 - x + 1$ that the sum may be x^3? That the sum may be $3x$? 15? 0?

30. What must be subtracted from $2x^2 - 3x + 1$ that the remainder may be x^3? $x^2 + 10$? 7? $a - x + 1$?

If $A = 4x^3 - 2x^2y + 3xy^2 + y^3$, $C = 3x^3 - x^2y + 2y^3$,
$B = 4x^3 - x^2y - xy^2 - 3y^3$, $D = x^3 - 2xy^2 + y^3$,

find the value of

31. $A - B + C - D$

32. $A - [B - (D + C)]$

33. $A - (B + C) + D$

34. $B + \{A - [C - D]\}$

35. By a diagram show that $-7 - (-3)$ and $-7 + 3$ have the same value.

36. In an election for two candidates, 32,544 votes were cast. The successful candidate had a majority of 2416 votes. How many votes did each candidate receive?

37. The Panama Canal is 49 miles long and the part of it through the lowlands is 4 miles more than 8 times the part through the hills (called the Culebra Cut). How long is each part?

38. How many examples in Exercise 2 (p. 13) can you now work at sight?

CHAPTER IV

MULTIPLICATION

47. Multiplication, at the outset, may be regarded as the process of finding the result (called the *product*) of taking one quantity (the *multiplicand*) as many times as there are units in another quantity (the *multiplier*).

The term *multiplication* has acquired a much broader meaning than this, which is sometimes expressed as follows: Multiplication is the process of finding a number (the product) which is obtained from a given number (the multiplicand) in the same way that another number (the multiplier) is obtained from unity.

Multiplication is useful as a means of shortening addition or subtraction. Later many other uses (often indirect) of multiplication will become evident.

MULTIPLICATION OF MONOMIALS

48. Multiplication of Coefficients. To multiply $4a$ by $3b$, we evidently take the product of all the factors of the multiplier and the multiplicand, and thus get $4 \times a \times 3 \times b$. Rearranging factors, we obtain as the product,

$$4 \times 3 \times a \times b \text{ or } 12ab.$$

Hence, in multiplying two monomials,

Multiply the coefficients to produce the coefficient of the product.

49. Multiplication of Literal Factors or Law of Exponents.

Ex. Multiply a^3 by a^2.

$$\begin{aligned} \text{Since } a^3 &= a \times a \times a \\ \text{and } a^2 &= a \times a \\ \therefore a^3 \times a^2 &= a \times a \times a \times a \times a = a^5. \end{aligned}$$

This may be expressed in the form

$$a^3 \times a^2 = a^{3+2} = a^5,$$

$$\text{or, in general, } a^m \times a^n = a^{m+n},$$

where m and n are positive whole numbers.

Hence, in multiplying the literal factors of a monomial,

Add the exponents of each letter that occurs in both multiplier and multiplicand.

$$\text{Ex. } 4a^2bc^3 \times 3a^3b^2x = 12a^5b^3c^3x.$$

50. Law of Signs. The law of signs in multiplication follows directly from the general law of signs as stated in Art. 31.

(1) + \$100 taken 5 times gives + \$500,
or, in general, a + quantity taken a + number of times gives a + result.

(2) \$100 of debts, that is, – \$100, taken 5 times gives – \$500,
or, in general, a – quantity taken a + number of times, gives a – quantity as a result.

(3) \$100 deducted 5 times, or \$100 × – 5, gives as the total amount of deduction – \$500,
or, in general, a + quantity taken a – number of times, gives a – quantity as a result.

(4) Deducting \$100 of debts 5 times from a man's possessions is the same as adding \$500 to his assets; that is,

$$-\$100 \times -5 = +\$500,$$

or, in general, a – quantity taken a – number of times gives a + quantity as a result.

We see from (1) and (4) that

$$\text{either } + \times +, \text{ or } - \times -, \text{ gives } +,$$

and from (2) and (3), that

$$\text{either } - \times +, \text{ or } + \times -, \text{ gives } -.$$

In brief, in multiplication

Like signs give plus; unlike signs give minus.

51. Multiplication of Monomials. Combining the results of Arts. 48, 49, and 50, we may express the method of multiplying one monomial by another as follows:

Multiply the coefficients together for a new coefficient;

Annex the literal factors, giving each factor an exponent equal to the sum of its exponents in the terms multiplied together;

Determine the sign of the result by the rule that like signs give $+$, *and unlike signs give* $-$.

Ex. 1. Multiply $5a^2bx^3$ by $-6ab^3y^2$.

The product is $-30\,a^3b^4x^3y^2$.

Ex. 2. Multiply $5a^{n+3}$ by $2a^{n-1}$.

Since $n+3$ and $n-1$, added, give $2n+2$, the product is $10a^{2n+2}$.

EXERCISE 15

	1.	2.	3.	4.	5.	6.
Multiply	-5	$-3a$	$3ab$	$30x^2y^2$	$4x$	$-5x$
by	4	-2	-5	-1	$-2x$	$-3x$

	7.	8.	9.	10.	11.	12.
Multiply	$3ax$	$-6xy^2$	$7ax$	$-5a^2b$	$6c^2d$	$-2x^2yz$
by	$-4ax$	$-7xy^2$	$-3ay$	$-4cd^2$	$-3cd^2$	$-8xy^2z^3$

	13.	**14.**	**15.**	**16.**	**17.**	**18.**
Multiply	$4x^2$	$\frac{1}{2}ax^3$	$.5x$	$2.1y^3$	$2\frac{1}{2}x^3$	$\frac{3}{4}x^2$
by	$.2x^3$	$\frac{2}{3}a^2x^3$	$.03x$	$.05y^2$	$\frac{3}{4}x$	$.5x$

	19.	**20.**	**21.**	**22.**	**23.**
Multiply	2^{n-1}	2^{n-1}	2^{n-1}	x^{n-1}	x^{n-2}
by	2^2	2^3	2	x^2	x^3

Verify Exs. 19–21 when $n = 4$. Also Exs. 22 and 23 when $n = 4$ and $x = 3$.

	24.	**25.**	**26.**	**27.**	**28.**
Multiply	a^2x^{n-1}	a^2x^{n-1}	a^2x^{n-3}	$-a^2x^{n-3}$	x^{2n}
by	x^3	$-ax^4$	$-a^3x^3$	ax^{n+1}	x^{3n}

29.	**30.**	**31.**	**32.**
$5(a+b)^3$	$3(a+b)^4$	$-6(a+b)$	$7(a+b)^{n-1}$
$2(a+b)^2$	$-(a+b)$	$-2(a+b)^3$	$3(a+b)^5$

33. Multiply 2^{n-2} by 2 and verify the result when $n = 4$.

34. Write out all the factors of $7a^3$. Of $(7a)^3$.

35. $(ab)^4$ is how many times as large as ab^4 when $a = 3$ and $b = 2$?

36. How many x's are there in the product of $5ax^3$ by $6a^2x^5$? How many a's?

37. How much money do five empty pocket-books contain? $5 \times 0 = ?$

38. Find the value of 7 times 0. Of $5a \times 0$. Of $0 \times 6x^2y^3$. Of $3(x + y) \times 0$.

If $a = 4$, $b = \frac{3}{2}$, $c = 0$, $x = 1$, and $y = 9$, find the value of

39. abc.

40. a^2c.

41. $5cxy^2$.

42. $a^2 + 3cy$.

43. $4c^2 + a$.

44. $(3c + x)^3$.

45. $\dfrac{ac + y}{3}$.

47. $\dfrac{2a + c(x + y)}{4}$.

46. $\dfrac{5ac^2 + 1}{x + y}$.

48. $\dfrac{5(x - 1) + 8}{2a}$.

49. How many of the examples in this Exercise can you work at sight?

50. How many examples in Exercise 3 (p. 19) can you now work at sight?

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

52. Utility of the Distributive Law in Multiplication; Rule. In arithmetic we have become familiar with the fact that, for instance, $5 \times 67 = 5(60 + 7) = 5 \times 60 + 5 \times 7$; and that this principle enables us to perform all multiplications by committing to memory only the products up to 9×9.

Similarly, in algebra, $a(b + c) = ab + bc$. This is called the *Distributive Law of Multiplication.* By use of this law, all multiplications in algebra can be performed as a multiplication of pairs of monomials.

Hence, to multiply any polynomial by a monomial,

Multiply each term of the multiplicand by the multiplier, and set down the results as a new polynomial.

Ex. Multiply $2a^3 - 5a^2b + 3ab^2$ by $-3ab^2$.

$$\begin{array}{rrcr} & 2a^3 - 5a^2b + 3ab^2 & = & 4 \\ & -3ab^2 & = & -12 \\ \hline \textit{Product} & -6a^4b^2 + 15a^3b^3 - 9a^2b^4 & = & -48 \end{array}$$

The check is obtained by letting $a = 1$, and $b = 2$.

EXERCISE 16

	1.	2.	3.	4.
Multiply	$2a + 3x$	$3x - 2y$	$4x^2y - xy^2$	$7ax - 4by$
by	$3ax$	$-5xy$	$2xy$	$-3abxy$

Multiply:

5. $8ac^2 - 3m^2n$ by $5an$.

6. $m - m^2 - 3m^3$ by $-7m^2n$.

7. $8x^2y - 5xy^2 - y^3$ by $3xy$.

8. $2x^n - 3x^{n-1}$ by x^3.

9. $3x^{n+1} + 7x^n$ by $-4x$.

10. $3x^{n-1} + 3x^{n-2}$ by x^2.

11. $3x^{5a} + 5x^{3a}$ by x^{4a}.

12. $2a^{5n} - 7a^{3n}$ by $-2a^{2n}$.

13.

$$\begin{array}{ll} \text{Multiply} & 2.5x^2 - 3.7x + .51 \\ \text{by} & .4x \\ \cline{2-2} \end{array}$$

14.

$$\begin{array}{l} \frac{3}{4}x - \frac{1}{2}x^2 - \frac{4}{5} \\ \frac{1}{2}x \\ \hline \end{array}$$

15.

$$\begin{array}{l} \frac{2}{5}ax^2 - \frac{1}{5}ax - \frac{3}{5}a \\ -\frac{1}{3}ax \\ \hline \end{array}$$

16.

$$\begin{array}{l} .4\,x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + .5x^4 \\ -.25x^3 \\ \hline \end{array}$$

17. What is the value of $7x - 5y$ times zero?

Multiply:

18. $5(a + b)^2 - 3(a + b) - 5$ by $2(a + b)$.

19. $7(x - y)^2 + 2(x - y) - 6$ by $3(x - y)^2$.

20. $2(3a + 2b)^2 - 5(3a + 2b) + 4$ by $4(3a + 2b)$.

21. Reduce $(7aaabb - 5aaabb) \times 6aabb$ to its simplest form. Compare the size of the result with that of the original expression.

(The following problems are mixed variations of **Types I, II**, and **III**.)

22. What number diminished by 19 equals 37?

23. What number increased by 19 equals 37?

24. What number diminished by 1.067 equals 4.5?

25. What number increased by twice itself and then by 24 equals 144?

26. What number increased by twice itself and then diminished by 24 equals 144?

27. What number increased by $\frac{1}{3}$ of itself and then by 20 equals 60?

28. What number diminished by $\frac{1}{4}$ of itself and then increased by 30 equals 90?

29. What number of dollars diminished by $\frac{1}{5}$ of itself and then by $30 equals $160.60?

30. If a number is multiplied by 3 and then diminished by 40, the result is 140. Find the number.

31. If 5 times a certain number is increased by 20.5, the result is 870. Find the number.

32. If five times a certain number is increased by 20.5, the result is equal to three times the number increased by 160. Find the number.

33. A man who died left $16,000 to his son and daughter. The share of his daughter, who had taken care of him in his illness, was $500 less than twice the share of the son. How much did each receive?

34. A cubic foot of iron and a cubic foot of aluminum together weigh 618 lb. If the weight of the iron is 14 lb. less than three times the weight of the aluminum, find the weight of each.

35. A baseball nine has played 54 games, and the number of games it has won is 3 less than twice the number it has lost. How many has it lost?

36. Of which type is each of the above problems (Exs. 22–35) an instance or a variation?

Multiply each member of the following equalities by -1 and solve:

37. $-2x-5=-x+4.$ **38.** $-4x-x=-6-9.$

39. Make up and work an example similar to Ex. 10. To Ex. 19.

40. Make up and work an example similar to Ex. 30. To Ex. 35.

41. How many of the examples in this Exercise can you work at sight?

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

53. Arranging the Terms of a Polynomial. The multiplication of polynomials is greatly facilitated by arranging the terms in each polynomial according to the powers of some letter, in either the ascending or descending order.

Thus, $5x^2+3-x+x^4-7x^3$, arranged according to the ascending powers of x, becomes

$$3-x+5x^2-7x^3+x^4.$$

Also, $a^4+b^4-4a^2b^2-5a^3b$, arranged according to the descending powers of a, becomes

$$a^4-5a^3b-4a^2b^2+b^4.$$

54. Multiplication of Polynomials. By a double use of the Distributive Law:

$$\begin{aligned}(a+b)(c+d)&=a(c+d)+b(c+d)\\&=ac+ad+bc+bd\end{aligned}$$

We see that a similar result is obtained, no matter how many terms occur in each polynomial.

Therefore, to multiply two polynomials,

Arrange the terms of the multiplier and the multiplicand according to the ascending or descending powers of the same letter;

Multiply each term of the multiplicand by each term of the multiplier;

Add the partial products thus obtained.

Ex. 1. Multiply $2x - 3y$ by $3x + 5y$.

The terms as given are arranged in order.

The most convenient way of adding partial products is to set down similar terms in columns, thus:

$$\begin{array}{lrrrrr}
 & 2x & -\ 3y & & = & -1 \\
 & 3x & +\ 5y & & = & 8 \\
\hline
\textit{Partial products}\left\{\begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{r} 6x^2 \\ \ \end{array} & \begin{array}{r} -\ 9xy \\ +\ 10xy \end{array} & \begin{array}{r} \ \\ -\ 15y^2 \end{array} & & \\
\hline
\textit{Product} & 6x^2 & +\ xy & -\ 15y^2 & = & -8
\end{array}$$

The *check* is obtained by letting $x = 1$ and $y = 1$ (or $x = y = 1$). Note that this method checks only the signs and coefficients, not the letters or their exponents. Mistakes in letters and exponents, however, are rare in comparison with mistakes in signs and coefficients. A convenient check for all elements in the process is obtained by letting $x = y = 2$. A useful check on the letters and exponents in many examples is given in Art. 56.

Ex. 2. Multiply $2x - x^3 + 1 - 3x^2$ by $2x + 3 - x^2$.

Arrange the terms in both polynomials according to the ascending powers of x. (Why is the ascending order chosen rather than the descending?)

$$\begin{array}{lrrrrrrrr}
 & 1 & +\ 2x & -\ 3x^2 & -\ x^3 & & & = & -1 \\
 & 3 & +\ 2x & -\ x^2 & & & & = & 4 \\
\hline
 & 3 & +\ 6x & -\ 9x^2 & -\ 3x^3 & & & & \\
 & & +\ 2x & +\ 4x^2 & -\ 6x^3 & -\ 2x^4 & & & \\
 & & & -\ x^2 & -\ 2x^3 & +\ 3x^4 & +\ x^5 & & \\
\hline
\textit{Product} & 3 & +\ 8x & -\ 6x^2 & -\ 11x^3 & +\ x^4 & +\ x^5 & = & -4
\end{array}$$

Now multiply the two polynomials together with their terms in the order as first given. This will show you the advantage of arranging the terms in order before multiplying.

Ex. 3. Multiply $a^2+b^2+c^2+2ab-ac-bc$ by $a+b+c$.

Arranging the terms according to powers of a,

$$\begin{array}{lllllllllll}
a^2 & +2ab & -ac & +b^2 & -bc & +c^2 & & & & & =3\\
a & +b & +c & & & & & & & & =3\\
\hline
a^3 & +2a^2b & -a^2c & +ab^2 & -abc & +ac^2 & & & & & \\
 & +a^2b & & +2ab^2 & -abc & & +b^3 & -b^2c & +bc^2 & & \\
 & & +a^2c & & +2abc & -ac^2 & & +b^2c & -bc^2 & +c^3 & \\
\hline
a^3 & +3a^2b & & +3ab^2 & & & +b^3 & & & +c^3 & =9
\end{array}$$

55. Degree of a Term; Homogeneous Expressions. The degree of a term is determined by the number of literal factors which the term contains. Hence, the degree of a term is equal to the sum of the exponents of the literal factors in the term.

Thus, $7a^3bc^2$ is a term of the 6th degree, since the sum of the exponents in it is $3+1+2$, or 6.

The *degree of an algebraic expression* is the same as the degree of that term in the expression which has the highest degree.

Thus, $7x^3+3x^2y^2+y$ is of the 4th degree.

A **homogeneous polynomial** is a polynomial of which all the terms are of the same degree.

Thus, $5a^2b-b^3+ab^2$ is a homogeneous polynomial, since each of its terms is of the 3d degree.

56. Multiplication of Homogeneous Polynomials. If two monomials are multiplied together, the degree of the product must equal the sum of the degrees of the multiplier and the multiplicand.

For instance, in Ex. 3, above, the multiplicand is of the 2d degree and the multiplier is of the 1st degree, and are both homogeneous. Their product is seen to be homogeneous and of the 3d degree.

The fact that *the product of two homogeneous expressions must also be homogeneous* affords a partial test of the accuracy of the work.

If, for instance, in Ex. 3, p. 67, a term of the 5th degree, such as $5a^3b^2$, had been obtained in the product, it would have been at once evident that a mistake had been made in the work.

57. Detached Coefficients; Symmetrical Expressions. The process of multiplying algebraic expressions may often be further abbreviated by using only the signs and coefficients of terms, omitting the letters and their exponents (see Appendix, p. 466).

EXERCISE 17

Multiply and check each result:

1. $x - 4$ by $2x + 1$.
2. $x - 3$ by $3x + 2$.
3. $2x + 5$ by $x - 7$.
4. $3x - 4y$ by $4x - 3y$.
5. $7x^2 - 5y^2$ by $4x^2 + 3y^2$.
6. $5xy + 6$ by $6xy - 7$.
7. $4a^2 - b^2c$ by $8a^3c + 2ab^2c^2$.
8. $11x^3y - 7xy^3$ by $3x^2 + 2y^2$.
9. $a^2 - ab + b^2$ by $a + b$.
10. $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
11. $4x^3 - 3x^2 + 2x - 1$ by $2x + 1$.
12. $2x^2 - 3xy + 2y^2$ by $3x - 5y$.
13. $x^3 - 3x^2 + 2x - 1$ by $2x^2 + x - 3$.
14. $3x^2y - 4xy^2 - y^3$ by $x^2 - 2xy - y^2$.
15. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
16. $4x^3 - 3x^2 + 5x - 2$ by $x^2 + 3x - 3$.
17. $x^4 - 3x^2 + 5$ by $x^2 - x - 4$.
18. $x^3 - 3xy + y^3$ by $x^3 - 3xy - y^3$.

19. $a^2 - ab + b^2$ by $a^2 + ab + b^2$.

20. $4x^2 + 9y^2 - 6xy$ by $4x^2 + 9y^2 + 6xy$.

21. $x^4 - 7x^2y^2 + 6xy^3 - y^4$ by $x^3 - 2xy^2 + y^3$.

22. $x^3 - 6ax^2 + 12a^2x - 8a^3$ by $-x^2 - 4ax - 4a^2$.

23. $a^2 + b^2 + x^2 + 2ab - ax - bx$ by $a + b + x$.

24. $ab + cd + ac + bd$ by $ab + cd - ac - bd$.

25. $\frac{1}{2}a + \frac{1}{2}b$ by $\frac{1}{2}a - \frac{1}{2}b$.

26. $\frac{2}{3}x^2 - 4x + \frac{4}{9}$ by $\frac{3}{4}x + \frac{9}{2}$.

27. $.5a - .4b$ by $.2a - .3b$.

28. $1.8x^2 - 3.2x + .48$ by $2.5x + .5$.

29. $x^n + 2x^{n-1} + 3x^{n-2} - 2$ by $x - 2$.

30. $x^{n+1} - 3x^n + 4x^{n-1} - 5x^{n-2}$ by $x^n + 2x^{n-1}$.

31. $x^{n-4} - 2x^{n-3} + 3x^{n-2} - 4x^{n-1} + 5x^n$ by $2x^2 + 3x + 1$.

32. Multiply $3x - 5 + 4x^2 + x^3$ by $2x - 3 + x^2$ without changing the order of the terms. Now arrange the terms in each expression in descending order and multiply. About how much easier is the second process than the first?

33. Make up and work an example similar to Ex. 32.

Arrange the terms of the following in descending order of some letter, and multiply:

34. $4x - 3x^2 - 5 + 2x^3$ by $x + 4$.

35. $3x^2 - 5 - x$ by $x + 4x^2 - 2$.

36. $2x^3 + y^3 - 4xy^2 + 3x^2y$ by $y^2 + 3x^2 - 2xy$.

37. Which of the polynomials in Exs. 16–24 are homogeneous?

38. A number increased by 3 times itself and then by 40 equals 180. Find the number.

39. Separate 180 into two parts such that one part exceeds three times the other by 40.

SUG. Let x = the second part.

40. A number increased by $\frac{1}{2}$ of itself and then by 20 equals 95. Find the number.

41. Separate 95 into two parts such that one part exceeds $\frac{1}{2}$ the other part by 20.

42. A number increased by $\frac{2}{3}$ of itself and then diminished by 30 equals 70. Find the number.

43. Separate 70 into two parts such that one part exceeds $\frac{2}{3}$ of the other part by 30.

44. A number diminished by $\frac{2}{5}$ of itself and then increased by 30 equals 66. Find the number.

45. A number increased by .06 of itself and then by \$100 equals \$312. Find the number.

46. Separate 400 into two parts such that one part exceeds 3 times the other part by 60.

47. Separate \$1000 into two parts such that one part is smaller than 4 times the other part by \$100.

48. Of which type is each of the above problems (Exs. 38–47) an instance or a variation?

58. Multiplication Indicated by the Parenthesis; Simplifications. The parenthesis is useful in indicating multiplications or combinations of multiplications.

Thus, $(a - b + 2c)^2$ means that $a - b + 2c$ is to be multiplied by itself.

$(a - b + 2c)^3$ means that $a - b + 2c$ is to be taken as a factor three times and multiplied.

To perform the multiplication expressed by a power is to *expand* the power.

Again, $(a - b)(a - 2b)(a + b - c)$ means that the three factors, $a - b$, $a - 2b$, and $a + b - c$, are all to be multiplied together.

Also, $(a - 2x)^2 - (a + 2x)(a - 2x)$ means that $a + 2x$ is to be multiplied by $a - 2x$, and the product is to be subtracted from the product of $a - 2x$ by itself.

We *simplify* an expression in which multiplications are indicated by parentheses and exponents by performing the operations indicated and collecting terms.

Ex. Simplify $3(x - 2y)(x + 2y) - 2(x - 2y)^2$.

$$\begin{aligned} 3(x - 2y)(x + 2y) - 2(x - 2y)^2 & \\ &= 3(x^2 - 4y^2) - 2(x^2 - 4xy + 4y^2) \\ &= 3x^2 - 12y^2 - (2x^2 - 8xy + 8y^2) \\ &= 3x^2 - 12y^2 - 2x^2 + 8xy - 8y^2 \\ &= x^2 + 8xy - 20y^2 \quad \textit{Ans.} \end{aligned}$$

Check this result by letting $x = 1$ and $y = 2$.

EXERCISE 18

Find the product of

1. $(-a)(-a)(-a)(-a)(-a)$.

2. $(-1)(-1)(-1)(-1)(-1)(-1)$.

3. $(x - y)(x - y)(x + y)(x - y)(x + y)$ in parenthesis form.

4. Find the value of $(-2)^4$. Of a^n when $a = -1$ and $n = 7$.

Simplify by removing parentheses and collecting terms:

5. $x - 2(x + 1)$.

6. $(x - 2)(x + 1)$.

7. $2x + 3(5x - 4)$.

8. $(2x + 3)(5x - 4)$.

9. $7a - 3(4a - 8)$.

10. $9a + 5(3a + 4)$.

11. $3x(x - 2) - 2x(x - 3)$.

12. $(2x^2 - 3x + 1)^2$.

13. $(2a - 3b + 5)^2 - (2a + 3b - 5)^2$.

14. $(x - 5)^2 - (x + 5)^2$.

15. $3x - 2(3x^2 - 5x + 2)$.

16. $x - 2(x - 1)(x + 3)$.

17. $(x - 2)(x - 1)(x + 3)$.

18. $3a^2 - (a - 2b)(3a + 4b)$.

19. $(x - y - z)^2 - x(x - 2y + 2z)$.

20. $2x^2 - 3(x - 1)^2 + (x - 2)^2$.

21. $3x^2 - x(1 - x)(2 + x) + x^3$.

22. $2 - 3(x - 2)^2 - 2(3 - 2x)(1 + x)$.

23. $a^2 - [x(a - x) - a(x - a)] - x^2$.

24. $(x - 1)(x - 2) - (x - 2)(x - 3) + (x - 3)(x - 4)$.

25. $3(x - y)^2 - 2\{(x + y)^2 - (x - y)(x + y)\} + 2y^2$.

26. $x(x - y - z) - y(z - x - y) - z(z - y - x) - y^2$.

27. $3[(a + 2b)x + 2my] - 5[(m - c)y + bx] - 4[(x - a)a + cy]$.

28. $26ab - (9a - 8b)(5a + 2b) - (4b - 3a)(15a + 4b)$.

29. Multiply the sum of $(a - 2x)^2$ and $(2a - x)^2$ by $3a - 2(a - x)$.

30. Subtract $(x - 2y)^3$ from $x^3 - 8y^3$ and divide the remainder by $x - 2y$.

31. Find the value of $3 \times 0 + 4$. Of $8 - 7 \times 0$. Of $6 \times 0 \times 5 + 7$.

If $a = 3$, $b = 0$, $x = -2$, and $y = -5$, find the values of

32. $2ax$.

33. bx^3y.

34. $3x^2 + aby$.

35. $bxy - ax^2$.

36. $by^2 + 3x(x - y)$.

37. $4x^2 - abx(4x - y)$.

38. $3x - 5(2x + 3)$.

39. $2(x^2 + y) - aby + ax^3$.

40. $2(1 - 2x)^2 + (x + y)(a^2 + x)$.

41. $(x - 1)^2 - 3(x + 1)(x + 2) - x(x^2 - 2)(y - 2x)$.

42. $3a(a - 2x) - \{a - (a - 1)(x + 1) - (a + x)^2\} + 5ax$.

Find the value of

43. $(x + a)^2 - (x - a)^2$ when $x = 2a$.

44. $5(x + p)^2 - (x + p)(x - 2p)$ when $x = 3p$.

45. $3x^2 + 4x - 5(x - 1)^2$ when $x = ab$.

46. If $x = 2$ and $y = 1$, find the value of $(x + y)^3$. Also of $x^3 + y^3$.

47. From the sum of $2a + 5b$ and $3b - 5a$, subtract three times $a - 7b$, and verify the result when $a = 2$ and $b = 5$. Also when $a = 3$ and $b = -1$.

48. If a certain number is diminished by 24 and the result multiplied by 3, the final result will be 78. Find the number.

49. If a certain sum of money is increased by $150 and the result multiplied by 4, the final result will be $1000. What is the original sum of money?

50. Separate $1000 into two parts such that one part equals four times the sum of $150 and the other part.

51. Separate .0015 into two parts such that one part equals 3 times the sum of .0001 and the other part.

52. The sum of two fractions is $1\frac{2}{3}$, and the larger is three times the sum of the smaller and $\frac{1}{3}$. Find the fractions.

53. Separate $100 into two parts such that the sum of one part and $10 equals the other part.

54. Separate $100 into three parts such that 3 times the sum of $5 and one of the parts equals each of the other parts.

55. Separate $100 into four parts such that twice the sum of one part and $1 equals each of the other parts.

56. A man walked 15 miles, rode a certain distance, and then took a boat for twice as far as he had previously traveled. Altogether he went 120 miles. How far did he go by boat?

57. The sum of three numbers is 50. The first number is twice the second, and the third is 16 less than three times the second. Find the numbers.

58. Find five consecutive numbers whose sum is 3 less than 6 times the least of the numbers.

59. The difference between two numbers is 6, and if 3 is added to the larger, the sum will be double the less. Find the numbers.

60. Divide $4500 among two sons and a daughter so that each son gets $100 less than twice the daughter's share.

61. Find two numbers, whose difference is 14, such that the greater exceeds twice the less by 3.

62. The difference of the squares of two consecutive numbers is 43. Find the numbers.

63. Three boys together earned $98. If the second earned $11 more than the first, and the third $28 less than the other two together, how much did each earn?

64. Which of Exs. 48–63 are instances or variations of Type I? Of II? III?

65. Make up and work an example similar to Ex. 48. To Ex. 49.

66. Make up and work an example similar to Ex. 58. To Ex. 62.

67. How many examples in Exercise 6 (p. 29) can you now work at sight?

CHAPTER V

DIVISION

59. Division is the process of finding one factor when the product and the other factor are given.

The **dividend** is the product of the two factors, and hence it is the quantity to be divided by the given factor.

The **divisor** is the given factor.

The **quotient** is the required factor.

Thus, to divide $10xy$ by $5x$, we must find a quantity which, multiplied by $5x$, will produce $10xy$. The factor $5x$ is the divisor, $10xy$ is the dividend, and the other factor, or required quotient, is evidently $2y$.

The division of a by b may be indicated in each of the following ways:

$$b\underline{)a,} \quad a \div b, \quad \frac{a}{b}, \quad \text{or } a/b$$

60. General Principle. Division being the inverse of multiplication, the methods of division are obtained by inverting the processes used in multiplication.

Division of Monomials

61. Index Law for Division. If a^5 is to be divided by a^2, we have

$$\frac{a^5}{a^2} = \frac{a \times a \times a \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}} = a \times a \times a = a^3$$

Or, in general, $$\frac{a^m}{a^n} = a^{m-n},$$

where m and n are positive whole numbers.

62. The Law of Signs in Division is obtained by inverting the processes of multiplication.

Thus, in multiplication, if a and b stand for any positive quantities (see Art. 50, p. 59),

$$\left.\begin{array}{l} +a\times+b=+ab\\ +a\times-b=-ab\\ -a\times+b=-ab\\ -a\times-b=+ab\end{array}\right\}\begin{array}{l}\text{Hence, by}\\ \text{definition}\\ \text{of divi-}\\ \text{sion,}\end{array}\left\{\begin{array}{l} +ab\div+b=+a\ldots(1)\\ -ab\div-b=+a\ldots(2)\\ -ab\div+b=-a\ldots(3)\\ +ab\div-b=-a\ldots(4)\end{array}\right.$$

From (1) and (2) we see that *the division of like signs gives* $+$. From (3) and (4) we see that *the division of unlike signs gives* $-$. Hence, the law of signs is the same in division as in multiplication.

63. Division of Monomials. Combining the results obtained in Arts. 60, 61, and 62, we have the following method for the division of one monomial by another:

Divide the coefficient of the dividend by the coefficient of the divisor;

Obtain the exponent of each literal factor in the quotient by subtracting the exponent of each letter in the divisor from the exponent of the same letter in the dividend;

Determine the sign of the result by the rule that like signs give plus, and unlike signs give minus.

Ex. 1. Divide $27a^3b^4x^3$ by $-9a^2bx^3$.

$$\frac{+27a^3b^4x^3}{-9a^2bx^3}=-3ab^3 \textit{ Quotient}$$

since the factor x^3 in the divisor cancels x^3 in the dividend.

Ex. 2. Divide a^{2m-3} by a^{m-1}.

$$\frac{a^{2m-3}}{a^{m-1}}=a^{m-2} \textit{ Quotient}$$

Check the work in each of the above examples by multiplying the quotient by the divisor.

EXERCISE 19

Divide and check the result:

1. $15a$ by $-5a$.

2. $-3x^3$ by x.

3. $8a^2x^2$ by $-4ax^2$.

4. $-30x^3y^2$ by $-6x^2y$.

5. $-7xz^3$ by $7z^3$.

6. $21xy^2z$ by $-3xz$.

7. $18bc^3d^3$ by $-9c^3d$.

8. $-33x^5y^6z^7$ by $11xy^3z^5$.

9. $28x^2y^2z^3$ by $-14xy^2z^3$.

10. $-m^3n$ by $-m^3$.

11. $-3x^2$ by -1.

12. $-8ax$ by $\frac{1}{2}x$.

13. $16by^2$ by $-\frac{2}{3}by$.

14. $8mx$ by $.2x$.

15. $.4ax^2$ by $.8x^2$.

16. $.04ax$ by $.5ax$.

17. $2\frac{1}{2}x^3$ by $\frac{3}{4}x^2$.

18. $-\frac{1}{4}x^2$ by $.5x$.

19. $4\pi r^2$ by 2π. By r^2. By πr.

20. $\frac{1}{2}gt^2$ by gt. By $\frac{1}{4}g$. By $\frac{1}{2}t$.

21. $\frac{1}{2}mv^2$ by $\frac{2}{3}m$. By $.5v^2$. By $.25v$.

22. $20(x+y)^3$ by $-4(x+y)$. By $-2(x+y)^2$.

23. $-1.4(a-b)^5$ by $-7(a-b)^3$. By $-2(a-b)^4$.

24. a^{6n} by a^{2n}. By a^{3n}. $-a^n$.

25. $-6a^{n+3}$ by $2a^{n+1}$. By a^{n+2}. a^n. $-3a^{n-1}$. a^{2n-3}.

26. a^{2n+5} by a^{n+1}. By a^{2n+3}. a^{n-3}.

27. How many 2's are multiplied together in 2^{10}? In 2^4? In the quotient of $2^{10} \div 2^4$?

28. How many x's in x^{10}? In x^4? In the quotient of $x^{10} \div x^4$?

29. Divide 2^{n-1} by 2 and verify your result when $n = 5$. Treat $2^{n-1} \div 2^3$ in the same way.

30. If an empty box is divided by partitions into 5 equal parts, will each compartment of the box be empty?

31. What is the value of $0 \div 5$? Of $0 \div 7$? State the meaning of the latter in a manner similar to that used in Ex. 30.

32. Give the value of $0 \div 10$. Of $0 \div a$. Of $0 \div 2x$. Of $\frac{0}{7a}$, $\frac{0}{7ab}$, $\frac{0}{7abx}$, $\frac{0}{7a^2b^2x^3}$.

33. What is the value of $\frac{3ax}{7y}$ when $a = 0$? When $x = 0$?

If $a = 2$, $b = 3$, $c = 0$, $x = 1$, find the value of each of the following:

34. $\frac{bc}{a}$

35. $\frac{c(a + x)}{b}$

36. $\frac{c(2b - x)}{4a}$

37. $\frac{5ac}{b + x}$

38. What is a polynomial? A binomial? A monomial? Give two examples of each.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

64. Utility in the Distributive Law of Division; Rule. In arithmetic we have become familiar with the fact that, for instance,

$$\frac{65}{5} = \frac{50 + 15}{5} = \frac{50}{5} + \frac{15}{5} = 10 + 3 = 13;$$

and that this principle enables us to perform all divisions by committing to memory only the quotients up to $81 \div 9$.

Similarly, in algebra, divisions can be greatly simplified by the fact that

$$\frac{ac + bc}{} = \frac{ac}{} + \frac{bc}{} = a + b$$

This is called the *Distributive Law of Division.*

Hence, to divide a polynomial by a monomial,

Divide each term of the dividend by each term of the divisor, and connect the results by the proper signs.

Ex. 1. Divide $12a^3x - 10a^2y + 6a^4z^2$ by $2a^2$.

$$\begin{array}{r|l} 2a^2) & 12a^3x - 10a^2y + 6a^4z^2 \\ \hline & 6ax - 5y + 3a^2z^2 \quad \textit{Quotient} \end{array}$$

Ex. 2. Divide $6a^{3n+3} - 4a^{2n+2} - 2a^{5n-3}$ by $2a^{n-1}$.

$$\begin{array}{r|l} 2a^{n-1}) & 6a^{3n+3} - 4a^{2n+2} - 2a^{5n-3} \\ \hline & 3a^{2n+4} - 2a^{n+3} - a^{4n-2} \quad \textit{Quotient} \end{array}$$

Check the work in each of the above examples by *multiplying the quotient by the divisor.*

EXERCISE 20

Divide and check:

1. $x^3 - 3x^2$ by $-x$.
2. $20x^2 - 8xy$ by $4x$.
3. $4ab^2 - 6a^2bc$ by $-2ab$.
4. $-3x^3 + 7x^2 - x$ by $-x$.
5. $15x^3y - 10x^2y^2 - 5xy^3$ by $5xy$.
6. $-m - m^2 + m^3 - m^4$ by $-m$.
7. $14x^3y^2z - 21xy^2z^3 + xyz$ by $-xyz$.
8. $-3x^2 - 2x + 5$ by -1.
9. $.6x^2 - .12x + 9$ by $-.3$.
10. $.02a^2 - .04ab - .8b^2$ by $.5$.
11. $\frac{1}{2}x^2 - \frac{2}{3}x - \frac{5}{2}$ by $-\frac{2}{3}$.
12. $\frac{2}{3}a^4b^2 - \frac{1}{6}a^3b^3 - \frac{1}{3}a^2b^4$ by $-\frac{3}{2}ab$.

13. $9x^{3n} - 6x^{2n} + 12x^n$ by $-3x^n$.

14. $-4x^{2n+1} + 10x^{2n+2} - 6x^{n+2}$ by $2x^{2n}$.

15. $x^{n+3} - 2x^{n+2} + 3x^{n+1} + x^n$ by x^{n-1}.

16. $8x^{m+2} - 16x^{m+1} - 4x^m - 12x^{m-1}$ by $-4x^{m-2}$.

17. $9x^{2n-2} - 6x^{2n-1} + 12x^{2n} - 3x^{2n+1}$ by $3x^{n-1}$.

18. $10(a + b)^2 - 8(a + b)$ by $-2(a + b)$.

19. $.5(x - y)^4 - .15(x - y)^3$ by $.5(x - y)^2$.

20. $(a + b)x - (a + b)y$ by $(a + b)$.

21. $(a - b)x + (a - b)y$ by $(a - b)$.

22. $x(x + 1) + (x + 1)$ by $(x + 1)$.

23. $\frac{1}{3}$ of a number added to twice the number gives 210. Find the number.

24. $\frac{2}{3}$ of a number added to 5 times the number gives 340. Find the number.

25. $\frac{2}{3}$ of a number added to $\frac{1}{2}$ the number gives 140. Find the number.

26. The difference between $\frac{2}{3}$ and $\frac{1}{2}$ of a certain number is 14. Find the number.

27. What number increased by .06 of itself gives 318?

28. What sum of money at simple interest for one year at 6% will amount to $318?

29. What number increased by .15 of itself will amount to 690?

30. What sum of money at simple interest at 5% will amount to $690 in 3 years?

31. For every nickel which a girl put in her savings bank her father put in a dime. If her bank contained $18.75 at the end of one year, how many nickels did the girl save in that time?

32. For every dime that a boy spent for books, his father gave him a quarter to spend for the same purpose. If he spent $52.50 in all, how much did his father give him?

33. A purse contains $10.50 in dollar bills and quarters, but there are twice as many quarters as bills. How many are there of each?

34. How can $2.25 be paid in 5 and 10 cent pieces so that the same number of each is used?

35. How can $5.95 be paid in dimes and quarters using the same number of each?

In the following equations divide each member by -1 and solve, checking each result:

36. $-1-3x=-x-5$

37. $-5x-8-x=-7x+1$

38. $-x-(2x-1)=-5$

39. $-7x-5=-3x+4$

40. How many of Exs. 23–35, pp. 81–82, belong to Type I? To Type II? III?

41. Make up and work an example similar to Ex. 31. To Ex. 36.

42. Make up and work an example similar to Ex. 13. To Ex. 18.

43. How many of the examples in this Exercise can you work at sight?

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

65. **General Method.** The method of dividing one polynomial by another is to arrange the polynomials according to the ascending or descending powers of some one letter,

and then, in effect, to separate the dividend into partial dividends, which are divided in succession by the divisor.

Ex. 1. Divide $6x^4 + 7x^3 - 3x^2 + 11x - 6$ by $2x^2 + 3x - 2$.

We divide the first term of the dividend, $6x^4$, by the first term of the divisor, $2x^2$, obtaining the quotient $3x^2$. Multiplying this quotient, $3x^2$, by the entire divisor, we obtain the first partial dividend. If we subtract this from the entire dividend and divide the remainder by $2x^2$, we have a process like the following:

$$
\begin{array}{rrrrr|l}
\multicolumn{5}{r|}{\overbrace{6x^4 + 7x^3 - 3x^2 + 11x - 6}^{\textit{Dividend}}} & \overbrace{2x^2 + 3x - 2}^{\textit{Divisor}} = 15 \div 3 \\
\cline{6-6}
6x^4 & + 9x^3 & - 6x^2 & & & \underbrace{3x^2 - x + 3}_{\textit{Quotient}} = 5 \\
\cline{1-3}
 & - 2x^3 & + 3x^2 & + 11x & - 6 & \\
 & - 2x^3 & - 3x^2 & + 2x & & \\
\cline{2-5}
 & & 6x^2 & + 9x & - 6 & \\
 & & 6x^2 & + 9x & - 6 & \\
\cline{3-5}
\end{array}
$$

A quick *check* on the parts of the work in which errors are most likely to be made is obtained by letting $x = 1$, as is done in the solution above. A more complete check is obtained by finding the product of the divisor and quotient and noting whether the result equals the dividend.

Now state the process of dividing a polynomial by a polynomial as a general rule.

Ex. 2. Divide $31ab^3 - 20b^4 - 10a^2b^2 + 6a^4 - a^3b$ by $3a^2 - 5b^2 + 4ab$.

$$
\begin{array}{rrrrr|l}
6a^4 & - a^3b & - 10a^2b^2 & + 31ab^3 & - 20b^4 & 3a^2 + 4ab - 5b^2 = 6 \div 2 \\
\cline{6-6}
6a^4 & + 8a^3b & - 10a^2b^2 & & & 2a^2 - 3ab + 4b^2 = 3 \\
\cline{1-4}
 & - 9a^3b & & + 31ab^3 & & \\
 & - 9a^3b & - 12a^2b^2 & + 15ab^3 & & \\
\cline{1-5}
 & & + 12a^2b^2 & + 16ab^3 & - 20b^4 & \\
 & & + 12a^2b^2 & + 16ab^3 & - 20b^4 & \\
\cline{3-5}
\end{array}
$$

Is the dividend homogeneous? The divisor? The quotient?

Ex. 3. Divide $x^3 + y^3 + z^3 + 3x^2y + 3xy^2$ by $x + y + z$.

Arranging terms according to the descending powers of x,

$$
\begin{array}{l|l}
x^3 + 3x^2y + 3xy^2 + y^3 + z^3 & \underline{x + y + z} \qquad\qquad\qquad\qquad = 9 \div 3 \\
x^3 + x^2y + x^2z & x^2 + 2xy - xz + y^2 + z^2 - yz = 3 \\
\cline{1-1}
\quad + 2x^2y - x^2z + 3xy^2 + y^3 + z^3 & \\
\quad + 2x^2y + 2xy^2 + 2xyz & \\
\cline{1-1}
\qquad - x^2z + xy^2 - 2xyz + y^3 + z^3 & \\
\qquad - x^2z - xz^2 - xyz & \\
\cline{1-1}
\qquad\quad xy^2 + xz^2 - xyz + y^3 + z^3 & \\
\qquad\quad xy^2 + y^3 + y^2z & \\
\cline{1-1}
\qquad\qquad + xz^2 - xyz - y^2z + z^3 & \\
\qquad\qquad + xz^2 + yz^2 + z^3 & \\
\cline{1-1}
\qquad\qquad\quad - xyz - y^2z - yz^2 & \\
\qquad\qquad\quad - xyz - y^2z - yz^2 & \\
\cline{1-1}
\end{array}
$$

The process of algebraic division may often be abbreviated by the use of detached coefficients. (See Appendix, p. 466.)

EXERCISE 21

Divide and check each result:

1. $3x^2 + 7x + 2$ by $x + 2$.
2. $6x^2 + 7x + 2$ by $3x + 2$.
3. $12x^2 + xy - 20y^2$ by $3x + 4y$.
4. $3x^2 + x - 14$ by $x - 2$.
5. $6x^2 - 31xy + 35y^2$ by $2x - 7y$.
6. $12a^2 - 11ac - 36c^2$ by $4a - 9c$.
7. $-15x^2 + 59x - 56$ by $3x - 7$.
8. $44x^2 - xy - 3y^2$ by $11x - 3y$.
9. $a^2 - 4b^2$ by $a - 2b$.
10. $x^3 - y^3$ by $x - y$.
11. $27x^3 + 8$ by $3x + 2$.
12. $9x^2 - 49$ by $3x + 7$.
13. $125 - 64x^3$ by $5 - 4x$.
14. $8a^3x^3 + y^6$ by $2ax + y^2$.

15. $2x^3 - 9x^2 + 11x - 3$ by $2x - 3$.

16. $35x^3 + 47x^2 + 13x + 1$ by $5x + 1$.

17. $6a^3 - 17a^2x + 14ax^2 - 3x^3$ by $2a - 3x$.

18. $4y^4 - 18y^3 + 22y^2 - 7y + 5$ by $2y - 5$.

19. $c^5 + c^4x + c^3x^2 + c^2x^3 + cx^4 + x^5$ by $c + x$.

20. $11x - 8x^2 + 5x^3 - 20 + 2x^4$ by $x + 4$.

21. $4x + 6x^5 + 3x^2 - 11x^3 - 4$ by $3x^2 - 4$.

22. $- x^3y - 11xy^3 - 2x^2y^2 + 6x^4 - 6y^4$ by $2x - 3y$.

23. $4y^3 + 6x^6 - 13x^4y$ by $3x^2 - 2y$.

24. $x^4 - 16y^4$ by $x - 2y$.

25. $x^5 + 32y^5$ by $x + 2y$.

26. $x^6 - y^6$ by $x + y$.

27. $256x^8 - y^8$ by $4x^2 - y^2$.

28. $9x - 18x^3 + 8x^4 - 13x^2 + 2$ by $4x^2 + x - 2$.

29. $10 - x^3 - 27x^2 + 12x^4 - 3x$ by $x + 4x^2 - 2$.

30. $22x^2 - 13x^3 + 10x^5 - 18x^4 + 5x - 6$ by $x + 5x^2 - 2$.

31. $14x^2y^3 - 16x^3y^2 + 6x^5 + y^5 + 5x^4y - 6xy^4$ by $3x^2 + y^2 - 2xy$.

32. $5a^4b - 3a^3b^2 - a^2b^3 + 3a^5 - 4b^5$ by $a^2 + 3ab + 2b^2$.

33. $x^3 - y^3 + z^3 - xyz - 2x^2z + 2yz^2$ by $x - y - z$.

34. $c^3 + d^3 + n^3 - 3cdn$ by $c + d + n$.

35. $y^6 - 2y^3 + 1$ by $y^2 - 2y + 1$.

36. $2x^6 + 1 - 3x^4$ by $1 + 2x + x^2$.

37. $6x^2y^2 - 6y^2z^2 - 6x^2z^2 - 13xyz^2 - 5xy^2z$ by $3xy + 2yz + 3xz$.

38. $x^5 - 39x + 15 - 2x^3$ by $3x^2 + 6x + x^3 + 15$.

39. $4x^6 - 9x^4 + 25 - 14x^3 - x^2$ by $2x^3 - x - 5 + 3x^2$.

40. $\frac{1}{4}a^2 - \frac{1}{4}b^2$ by $\frac{1}{2}a + \frac{1}{2}b$.

41. $\frac{1}{9}x^2 - \frac{1}{16}y^2$ by $\frac{1}{3}x - \frac{1}{4}y$.

42. $\frac{1}{8}a^3 - \frac{1}{27}b^3$ by $\frac{1}{2}a - \frac{1}{3}b$.

43. $\frac{3}{4}x^3 - \frac{11}{6}x + \frac{2}{3}$ by $\frac{3}{2}x - 2$.

44. $.16x^2 - .25y^2$ by $.4x + .5y$.

45. $2.88x^3 + 10.86x - 19.2$ by $1.2x^2 + 1.5x + 6.4$.

46. $6x^{2n+1} - 13x^{2n} + 6x^{2n-1}$ by $3x^{n+1} - 2x^n$.

47. $12x^{4n} + 13x^{3n} - x^n$ by $3x^n + 1$.

48. $4x^{n+3} + 5x^{n+2} - x^{n+1} - x^n + x^{n-1}$ by $x^2 + 2x + 1$.

49. $6x^{n+1} - 5x^n - 6x^{n-1} + 13x^{n-2} - 6x^{n-3}$ by $2x^2 - 3x + 2$.

50. In Ex. 20 try to divide without arranging the terms of the dividend either in ascending or descending order.

51. What is the value of $\frac{7xy}{x+y}$ when $x = 0$? When $y = 0$?

52. Divide $x^5 + y^5$ by $x - y$ to 5 terms and note the remainder.

53. Divide 1 by $1 - x$ to 4 terms and note the remainder.

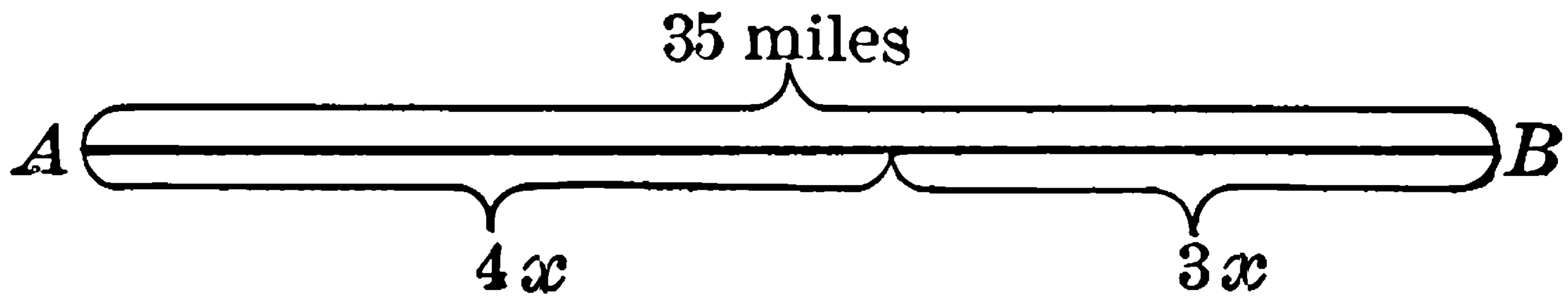
54. Divide 1 by $1 - ax$ to 3 terms.

55. If a boy walks at the rate of 3 miles an hour, how far will he walk in 5 hours? In a hours? In x hours? In $x + 2$ hours?

56. A boy starts at a given time and walks 5 hours. Another boy then starts and rides a bicycle x hours until he overtakes the first boy. How many hours does the second boy ride? How many if the first boy has a start of a hours? Of y hours?

57. Two men A and B start from places 35 miles apart and walk toward each other at the rate of 4 miles and 3 miles an hour respectively. How many hours will it be before they meet?

Sug. In forming an equation, it is an aid to diagram a problem of this kind:



If the two men start at the same time and walk toward each other until they meet, they must travel the same number of hours.

Let x = the number of hours each man travels.
Then $4x$ = number of miles A travels.
$3x$ = number of miles B travels.
$4x + 3x = 35$ (Art. 15, 1)
$7x = 35$
$x = 5$, *no. hours before they meet.*

Check. $4x = 20$, distance A travels.
$3x = 15$, distance B travels.
$20 + 15 = 35$

In working Exs. 58–72, draw a diagram as an aid in each solution:

58. Two men, A and B, start from places 42 miles apart and walk toward each other, at the rate of 4 and 3 miles per hour respectively. How many hours will it be before they meet?

59. Make up and work an example similar to Ex. 58.

60. Two bicyclists, A and B, start respectively from New York and Philadelphia, 90 miles apart, and ride toward each other. A rides 8, and B, 12 miles per hour. How long and how far will A ride before meeting B?

61. Boston is 234 miles from New York. If two automobiles start from the two cities at the same time and travel toward each other at the rate of 12 and 14 miles per hour respectively, how far will each go before they meet?

62. Make up and work a similar example concerning trains which travel between New York and Chicago, which are 912 miles apart.

63. One boy starts at a certain time from New York on a bicycle and travels toward Philadelphia at the rate of 8 miles an hour. One hour later another boy starts from Philadelphia and goes toward New York at the rate of 6 miles an hour. How long before they will meet?

64. New York and Washington are 228 miles apart. A train starts from New York at a given time and goes at the rate of 26 miles an hour, and two hours later a train starts from Washington and proceeds at the rate of 34 miles an hour. How long before they will meet?

65. Make up and work an example similar to Ex. 64 concerning trains which travel between Cincinnati and New Orleans, which are 830 miles apart.

66. Two boys start at the same place and travel in opposite directions on bicycles at the rate of 8 miles and 10 miles an hour. How long before they will be 108 miles apart?

67. If they travel in the same direction, how long before they will be 16 miles apart?

68. Two boys start from New York and Philadelphia at the same time and travel toward each other until they meet. If one goes twice as fast as the other and they meet in $7\frac{1}{2}$ hours, what is the rate per hour of each boy?

69. If in Ex. 68 one boy went 5 miles an hour faster than the other, and they met in 6 hours, what was the rate of each?

70. Make up and work an example similar to Ex. 68 concerning automobiles traveling between New York and Washington.

71. Make up and work an example similar to Ex. 69 concerning railroad trains traveling between New York and Buffalo, which are 440 mi. apart.

72. A set out from a town, P, to walk to Q, 45 miles distant, an hour before B started from Q toward P. A walked at the rate of 4 miles an hour, but rested 2 hours on the way; B walked at the rate of 3 miles an hour. How many miles did each travel before they met?

73. How many examples in Exercise 10 (p. 45) can you now work at sight?

EXERCISE 22

REVIEW

1. Express the following in as few terms as possible: $3.2x^2 - 2.5xy + .16y^2 + 1.5x^2 - .8y^2 - .32xy + .4y^2 - 1.5x^2 + .4xy$.

2. Subtract $.15a^2 + .3b^2 - 2.5ab$ from $-7a^2 - 4ab - 1.5b^2$.

3. Add $2\frac{1}{5}p^2 - 1.5p + \frac{7}{2}$, $.75p^2 + \frac{3}{2}p - .4$, $\frac{3}{4} - 6\frac{3}{4}p^2 - .5p$.

4. Simplify $3.2x^2 - [.8x^2 + (3.5x - \frac{1}{2} - .2x^2) - 1.5 - 3x]$.

5. Solve $.3x - 4 = .2x + .5$.

6. What is the root of an equation? How do you check your solution of an equation? Check Ex. 5.

7. Simplify $5x - 3(x - 2)(x + 7) + 3(x - 2)^2$.

If $a = 0$, $b = 1$, $c = 4$, $x = -2$, find the value of

8. $a(b + c) - 3x$.

9. $(c + 2x)(b - a) - 3(x + 4)(x + 5)$.

10. $\dfrac{3a + 5(2 + x)}{b + c}$.

11. Multiply $x - 2 + 4x^2$ by $2x^2 - 1 - 5x$.

12. Divide $x^4 + 3 - 6x^2 + x^5 + 8x - 11x^3$ by $2x - x^2 + 3$.

13. Find three consecutive numbers whose sum is 33.

14. In a certain kind of concrete, twice as much sand is used as cement, and twice as much gravel as sand. How many pounds of each are used in making 2800 lb. of concrete?

15. The record time for the 100 yd. swim at a certain date was $55\frac{2}{5}$ sec. This was $7\frac{2}{5}$ sec. more than 5 times that for the 100-yd. dash. What was the record time for the latter?

16. Solve Ex. 15 without using x to represent the unknown number. How much of the labor of writing out the solution is saved by the use of x? Is there any other advantage in using x to solve problems?

17. What is the dividend when the quotient is $x^3 + 2x^2 + 7x + 20$, the remainder $62x + 59$, and the divisor $x^2 - 2x - 3$?

18. What is the divisor if the quotient is $x^3 + 3x$, the dividend $x^5 - 8$, and the remainder $9x - 8$?

19. If $x = -\frac{2}{3}$ and $y = -\frac{3}{2}$, find the value of

$$(3x - 2y)^2 (9x^2 + 4y^2) - 6(y - x) \sqrt{6xy(x + 2y^2 + \tfrac{1}{3})}.$$

20. Add a to b. Also add $3a - 5b$ to $4c + 7d$.

21. Subtract $3x - 2y + z$ from -7. From $-a$. From b.

22. Subtract $2a - 3b$ from 0. Also 5 from 0.

23. Can $3 + 2ab$ be united in a single term? Give a reason.

24. The product of an even number of negative factors has what sign? Of an odd number of negative factors? Give an example using not less than five factors.

25. Express the following in a simpler form: $5aaa(x - y)(x - y)(x - y)(x - y)$.

26. If a boy's mark on each of three recitations is 0, what is his average on the recitations? Give the value of $0 + 0 + 0$. Of 3×0. Of $\frac{0}{3}$.

27. Find the value of $8 \times 0 - 5 + \frac{0}{7}$.

28. Form the power whose base is 5 and exponent 2. Also the power whose base is 2 and exponent 5. Find the difference in value between these two powers.

29. Find the value of each of the following products and verify each result for the values $x = 2, a = 3, b = 1$.

(1) $x^a.x^{2a}$ (2) $x^{2a+b}.\ x^{a-b}$ (3) $x^{a+b}.\ x^{a-b}$

30. From the product of $3x^2 - 2$ and $2x - 5$ subtract 7 times the product of x and $x - 2$.

31. Show on squared paper that $3 \times 4 + 5 \times 4 = 8 \times 4$. Also that $4 \times 5 + 7 \times 5 - 2 \times 5 = 9 \times 5$.

32. Multiply $\frac{2}{3}x^2 - ax + \frac{2}{3}a^2$ by $\frac{3}{4}x^2 + \frac{1}{2}ax + \frac{1}{3}a^2$.

33. Divide $x^{3n} - y^{3n}$ by $x^n - y^n$.

34. Divide $2 - x$ by $1 + x$ to five terms in the quotient.

35. Divide $[(x^2 - 2x - 1)(x - 1) + 2x^2 - 2x]$ by $[(x + 2)(x + 1) - (x^2 + 2x + 3)]$.

36. Multiply $3.2x^2 - 4.5xy + 1.8y^2$ by $1.5x - 3.5y$.

37. Divide $x^6 - 15$ by $x^2 + x - 1$ to five terms.

38. Divide $36x^2 + \frac{1}{9}y^2 + \frac{1}{4} - 4xy - 6x + \frac{1}{3}y$ by $6x - \frac{1}{3}y - \frac{1}{2}$.

39. Divide $2.4x^3 - 0.12x^2y + 4.32y^3$ by $1.5x + 1.8y$.

40. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Solve and verify

41. $(2x + 1)(x - 3) + 7 = x - 2(x - 4)(2 - x)$.

42. $7x - 2(x - 1)(2 - x) - 17 = x(3x + 7) - (x + 1)^2$.

Simplify:

43. $6z + [4z - \{8x - (2z + 4x) - 22x\} - 7x] - [7x + \{14z - (4z - 5x)\}]$.

44. $a^2(b - c) - b^2(a - c) + c^2(a - b) - (a - b)(a - c)(b - c)$.

45. What is the advantage of regarding a polynomial as made up of terms? (What is a polynomial? A term?)

46. What is the name of an expression containing two terms? Three terms?

47. Write a homogeneous expression containing three terms, and the letters x and y.

48. If $s = ar^{n-1}$, find the value of s when $a = 2$, $r = 3$, $n = 4$. Also when $a = 2$, $r = 1$, and $n = 5$. When $a = 3$, $r = \frac{1}{3}$, and $n = 5$.

49. Who first used the letters x, y, and z to represent unknown numbers in equations in algebra? (See p. 455.) Find out all you can about this man.

50. Give some of the other symbols that were used to represent numbers before the use of the three last letters of the alphabet was suggested.

51. Can you point out any advantages in the use of x, y, and z instead of the other symbols once used for the same purpose?

52. Find out, if you can, whether any other symbols than the last letters of the alphabet are now used to represent an unknown number in an equation? How many different symbols can be used for this purpose?

53. Who invented the parenthesis sign and when?

54. How many examples in Exercise 13 (p. 54) can you work at sight?

CHAPTER VI

EQUATIONS (*continued*)

66. The Equation, members of an equation, and transposition have already been explained. (See Arts. 42–45, pp. 52–53.)

67. A Root of an equation is a number which, when substituted for the unknown quantity, satisfies the equation; that is, reduces the two members of the equation to the same number.

Ex. If in the equation, $3x - 1 = 2x + 3$,
we substitute 4 in the place of x in each member,
we obtain $\quad 3x - 1 = 12 - 1 = 11$
$\quad 2x + 3 = 8 + 3 = 11$
The equation is satisfied. Hence, 4 is the root of the given equation.

68. The Degree of an Equation having One Unknown Quantity. If an equation contains only one unknown quantity, the degree of the equation (after the equation has been reduced to its simplest form) is determined by the highest exponent of the unknown quantity in the equation.

Thus, if x is the only unknown,

$2x + 1 = 5x - 8$ is an equation of the *first* degree
$ax = b^2 + cx$ is of the *first* degree.
$4x^2 - 5x = 20$ is of the *second* degree.
$3x^2 - x^3 = 6x + 8$ is of the *third* degree.

A **simple equation** is an equation of the first degree.

An equation of the first degree is also often termed a *linear equation*, for reasons which will be explained later. (See Art. 148.)

69. Identities and Conditional Equations. If we take the expression $(x - 2)(x + 2) = x^2 - 4$, and substitute $x = 1$, we obtain $-3 = -3$. The two members of the expression are found to be equal.

Similarly, they are found to be equal if we let $x = 2, 3, 4$, etc.; $0, -1, -2$, etc.; or any number. An expression having this characteristic is termed an *identity.*

An **identity** (or identical equation) is an equality whose two members are equal for all values of the unknown quantity (or quantities) contained in it.

A **conditional equation** is an equation which is true for only one value (or a limited number of values) of x. For the sake of brevity, a conditional equation is usually termed an *equation.*

The equations studied in Art. 42 (p. 52) and Exercise 13 (p. 54) are conditional equations.

Hence, the sign $=$ is used in two senses in elementary algebra, viz.: to indicate sometimes an equation, and sometimes an identity. The context enables us to decide readily which of these two meanings the sign $=$ has in any given case.

Later it will be found useful to use the mark $\equiv$ to indicate an identity, and $=$ to indicate a conditional equation, or equation proper.

70. The Aids in Solving an Equation, given in Art. 15, p. 18, stated more precisely, are as follows:

The roots of an equation are not changed if

1. *The same quantity is added to both members of the equation.*

2. *The same quantity is subtracted from both members of the equation.*

3. *Both members are multiplied by the same quantity or equal quantities* (provided the multiplier is not zero, or an expression containing the unknown).

4. *Both members are divided by the same quantity* (provided the divisor is not zero, or an expression containing the unknown).

Other principles similar to these are used later as aids in solving equations.

Transposition (see Art. 45, p. 52) is a short way of using Principles 1 and 2 of this article.

71. The Method of Solving a Simple Equation may now be stated as follows:

Clear the equation of parentheses by performing the operations indicated by them;

Transpose the unknown terms to the left-hand side of the equation, the known terms to the right-hand side;

Collect terms;

Divide both members by the coefficient of the unknown quantity.

Ex. Solve $x(x-2) = x(x+4) - 3(x-3)$. . . . . . (1)

Removing parentheses, $x^2 - 2x = x^2 + 4x - 3x + 9$

Transposing terms (Art. 70, 1, 2), $x^2 - x^2 - 2x - 4x + 3x = 9$

Collecting terms, $-3x = 9$ . . . . (2)

Dividing by -3 (Art. 70, 4), $x = -3$ *Root*

Check. $x(x-2) = -3(-3-2) = -3(-5) = 15$

$x(x+4) - 3(x-3) = -3(-3+4) - 3(-3-3)$

$= -3 - 3(-6) = -3 + 18 = 15$

EXERCISE 23

Solve the following; refer to each principle in Art. 70 as you use it, and check each answer:

1. $2x = 15 - 3x$.
2. $15 + 3x = 27$.
3. $4x - 11 = 29$.
4. $16x + 3 = 15x + 7$.
5. $14x - 10 = 12x - 3x$.
6. $3x - 7 = 14 - 4x$.
7. $2x - 7 = 8 + 5x$.
8. $2x - (x - 1) = 5$.
9. 2 ft. + x = 12 ft.
10. 7 in. + x = 2 ft.

11. $x^2 - x(x + 5) = x + 12.$ 13. $7(2 - 3x) = 2(7 - 8x).$

12. $2x - 3(x - 3) + 2 = 0.$ 14. $3 - 2(3x + 2) = 7.$

15. $(x - 8)(x + 12) - (x + 1)(x - 6) = 0.$

16. $5(x - 3) - 7(6 - x) + 3 = 24 - 3(8 - x).$

17. $3(x - 1)(x + 1) = x(3x + 4).$

18. $4(x - 3)^2 = (2x + 1)^2.$

19. $8(x - 3) - (6 - 2x) = 2(x + 2) - 5(5 - x).$

20. $5x - (3x - 7) - \{4 - 2x - (6x - 3)\} = 10.$

21. $x + 2 - [x - 8 - 2\{8 - 3(5 - x) - x\}] = 0.$

22. $2x(x - 5) - \{x^2 + (3x - 2)(1 - x)\} = (2x - 4)^2.$

23. $8x^2 + 13x - 2\{x^2 - 3[(x-1)(3 + x) - 2(x + 2)^2]\} = 3.$

24. $.25x - 2 = .2x + 3.$ 26. $\frac{5}{2}x + 6 = \frac{1}{2}x + 8.$

25. $1.6x - .7 = 1.5x - .3.$ 27. $\frac{7}{9}x - \frac{2}{3} = \frac{7}{3} - \frac{2}{9}x.$

28. $(.2x + .2)(.4x - .3) = (.4x - .4)(.2x + .3).$

29. What right have we to change the equation $3x = 15 - 2x$ to the form $3x + 2x = 15$?

30. If $2x - 3 = 5$, what right have we to transpose the -3, and to write the equation in the form $2x = 5 + 3$?

31. What is the advantage in being able to add the same number to both members of an equation? To transpose a term? To divide both members of an equation by the same number?

32. Determine which of the following are identities and which conditional equations, or equations proper:

(1) $(x + 3)(x - 3) = x^2 - 9.$

(2) $(x + 3)(x - 3) = (x + 1)(x - 2).$

(3) $(x + 2)(x - 1) = x^2 + x - 2.$

(4) $(x + 2)(x - 1) = x^2.$

33. Write an identity and an equation of which the first members are the same.

34. Prove that the sum of any three consecutive numbers equals three times the middle one of the numbers.

SUG. Let the three numbers be indicated by n, $n - 1$, and $n - 2$.

35. Find a similar result for the sum of five consecutive numbers. Of seven consecutive numbers.

36. Prove that the product of the sum and difference of any two numbers is equal to the square of the first, minus the square of the second. Illustrate by a numerical example.

SUG. Denote the two numbers by a and b.

37. Prove that the square of the sum of any two numbers equals the square of the first number, plus twice the product of the two numbers, plus the square of the second number. Illustrate by a numerical example.

38. State and prove a similar property for the square of the difference of two numbers.

39. Prove that if the sum of the cubes of two numbers is divided by the sum of the numbers, the quotient equals the square of the first number, minus the product of the first by the second, plus the square of the second.

40. State and prove a similar property of the difference of the cubes of two numbers.

Find the value of the letter in each of the following:

41. $3a - 2 = 7$.

42. $5 - 2b = 1$.

43. $5(c - 1) = 12 - c$.

44. $24 = 12 - 3p$.

45. $3(y - 4) = 5(2 - y)$.

46. $r - 3(r - 1) = 5$.

47. In $A = lw$, if $A = 42\frac{1}{2}$ and $l = 8\frac{1}{2}$, find w.

48. If $A = 48.36$ sq. ft. and $w = 6.2$ ft., find l.

49. Convert each of the two preceding examples into an example concerning areas.

50. If $V = lwh$, and $V = 504$, $l = 12$, and $h = 5$, find w.

51. Convert Ex. 50 into an example concerning volumes.

52. In $i = prt$, if $i = \$27$, $r = .05$, and $p = \$240$, find t.

53. Convert Ex. 52 into an example concerning interest.

54. So far as we know, who first used an equation to solve a problem? Give this first problem thus solved, and tell all you know about the document in which it was found. (See pp. 454 and 462.)

55. Form an equation whose root is 2 and which contains four terms.

56. Make up and work an example similar to Ex. 15. To Ex. 31. Ex. 46.

57. How many of the examples in this Exercise can you work at sight?

72. Solution of Problems. In solving problems, the student will find it necessary to study each problem carefully by itself, as no rule or method can be found which will cover all cases. The following general directions will, however, be found of service:

By study of the problem, determine what are the unknown quantities whose values are to be obtained;

Let x equal one of these unknown quantities;

State in terms of x all the other unknown quantities which are either to be determined or to be used in the process of the solution;

Obtain an equation by the use of a principle (such as, the whole is equal to the sum of its parts, or things equal to the same things are equal to each other);

Solve the equation, and find the value of each of the unknown quantities.

In solving problems it is especially important to note that we *let x = a definite number, not a vague quantity.*

Thus, in working Ex. 1 of Exercise 24

we do not let x = A's marbles,
nor x = what A has,
but let x = number of marbles A has.

73. Checking the Solution of a Written Problem. The best way of checking the result obtained by solving a problem is to observe whether the result obtained satisfies the conditions as originally stated in the language of the problem. (This method is better than that used in checking the example in Art. 46, p. 53.)

Thus, to check Ex. 18, p. 54: after the answers 9 and 4 have been obtained, we note that the difference of 9 and 4 is 5, and that the sum of 9 and 4 is 13. 9 and 4 thus satisfy the original conditions of the problem.

What is the advantage in this method of checking the solution of a problem?

EXERCISE 24.

ORAL

1. A has x marbles, and B has twice as many. How many has B? How many have both?

2. There are 100 pupils in a school, of which x are boys. How many are girls?

3. If I have x dollars, and you have three dollars more than twice as many, how many have you? How many have we together?

4. Two boys together solved a examples. One did x examples. How many did the other solve?

5. The difference between two numbers is 15, and the less is x. What is the greater? What is their sum?

6. If n is a whole number, what is the next larger number? The next less?

7. Write three consecutive numbers, the least being x. Write them if the greatest is y.

8. John has x dollars, and James has seven dollars less than three times as many. How many has James?

9. If I am x years old now, how old was I ten years ago? a years ago? How old will I be in c years?

10. A man bought a horse for x dollars, and sold it so as to gain a dollars. What did he receive for it?

11. A man sold a horse for \$200, and lost x dollars. What did the horse cost?

12. If a yard of cloth cost m dollars, what will x yards cost?

13. A boy rides a miles an hour; how far will he ride in c hours?

14. A bicyclist rides x yards in y seconds. How far will he ride in one second? In n seconds?

15. In how many hours can a boy walk x miles at a miles an hour?

16. A man has a dollars and b quarters. How many cents has he?

17. How many dimes in x dollars and y half-dollars?

18. I have x dollars in my purse and y dimes in my pocket. If I give away fifty cents, how much have I remaining?

19. By how much does 30 exceed x?

20. What number is 40 less than x? What number is x less than 40?

21. What number exceeds x by a? What number exceeds a by x?

22. By how much does $a + b$ exceed x?

23. How much did a girl have left if she had \$5 and spent 15¢? If she had a dollars and spent b cents?

24. A boy had a dollars, received b cents, and then spent c cents. How many cents did he have left?

25. What is the interest on a dollars at b per cent for c years?

26. Express algebraically the following statement: a divided by b gives c as a quotient and d as a remainder.

27. A man having x hours at his disposal, rode a hours at the rate of 8 miles an hour, and walked the rest of his time at the rate of 3 miles an hour. How far did he ride? How far did he walk?

EXERCISE 25

1. Separate $84 into two parts such that one part is three times as large as the other.

2. Separate $84 into two parts such that one part exceeds the other by $12.

3. Separate $84 into three parts such that the first part is twice as large as the second, and the second part is twice as large as the third.

4. A boy has three times as many marbles as his brother, and together they have 48; how many has each?

5. A and B pay together $100 in taxes; if A pays $22 more than B, what does each pay?

6. Two boys made $67.50 one summer by taking passengers on a launch. The boy who owned the launch received twice as large a share of the profits as the other boy. How much did each receive?

7. How many grains of gold are there in a gold dollar, if the gold dollar weighs 25.8 grains and 9 parts of the dollar are gold and 1 part copper?

8. A ball nine has played 64 games and won 12 more than it has lost. How many games has it won?

9. A man left $21,000 to his wife and four daughters. If the wife received three times as much as each daughter, how much did each receive?

10. If he had left $21,000 so that the wife received $10,000 more than each daughter, how much would each have received?

11. A cubic foot of water and a cubic foot of alcohol together weigh 112.5 lb. The alcohol weighs $\frac{4}{5}$ as much as the water. What is the weight of a cubic foot of each?

12. Find three consecutive numbers whose sum is 63.

13. In a certain grade of milk the other solids equal three times the weight of the butter fat, and the liquid part of the milk weighs 7 times as much as the solids. How many pounds of butter fat in 4800 lb. of milk?

14. The difference of the squares of two consecutive numbers is 43. Find the numbers.

15. At a certain date the record time for the quarter-mile run was 47 seconds, and 5 times the record time for the 100-yard dash exceeded the record time for the quarter-mile by 1 second. Find the record time for the 100-yard dash at this date.

16. The difference of two numbers is 13 and their sum is 35. Find the numbers.

17. John solved a certain number of examples, and William did 12 less than twice as many. Together they solved 96. How many did each solve?

18. Three boys earned together $98. If the second earned $11 more than the first, and the third $28 less than the other two together, how many dollars did each earn?

19. The sum of two numbers is 92, and the larger is 3 less than four times the less. Find the numbers.

20. The sum of three numbers is 50. The first is twice the second, and the third is 16 less than three times the second. Find the numbers.

21. A farmer paid $94 for a horse and cow. What did each cost, if the horse cost $13 more than twice as much as the cow?

22. Ex. 1 (p. 95) might be stated as a problem concerning an unknown number, thus: Twice a certain number equals 15 less three times the number. Find the number.

In like manner, convert Ex. 2 (p. 95) into a problem concerning an unknown number. Also Ex. 3. Ex. 8.

23. In reducing iron ore in a furnace, 7 times as many carloads of coke as of limestone are used, and 8 times as many carloads of iron ore as of limestone. If 800 carloads in all are used on a certain day, how many carloads of each is this?

24. One side of a triangle is twice as long as the shortest side. The third side exceeds the length of the shortest side by 12 yards. If the perimeter of the triangle is 360 yards, find each side.

25. A man spent $3.24 for coffee and sugar, buying the same number of pounds of each. If the sugar cost 5 cents a pound and the coffee 22 cents, how many pounds of each did he buy?

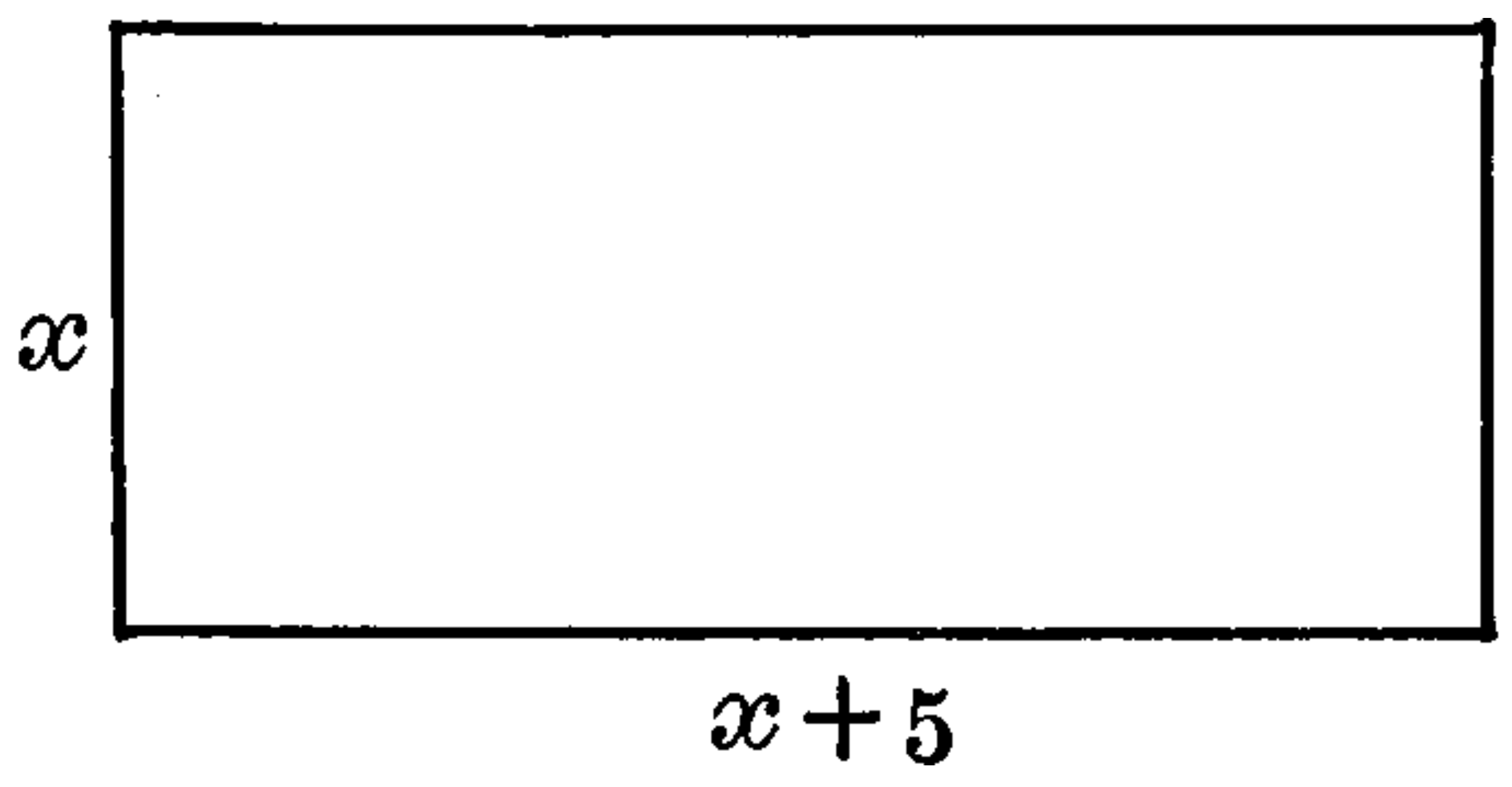
26. The distance from New York to Chicago is 912 miles. If this is 24 miles less than four times the distance from New York to Boston, find the latter distance.

27. On a certain railroad in a given year the receipts per mile were $3085. If the receipts per mile for freight exceeded those for passengers by $265, find the receipts per mile from each of these sources.

28. A man left $64,000 to his wife, daughter, and niece. To his daughter he left $4000 more than to his niece, and to his wife $8000 more than to his daughter and niece together. How much did he leave to each?

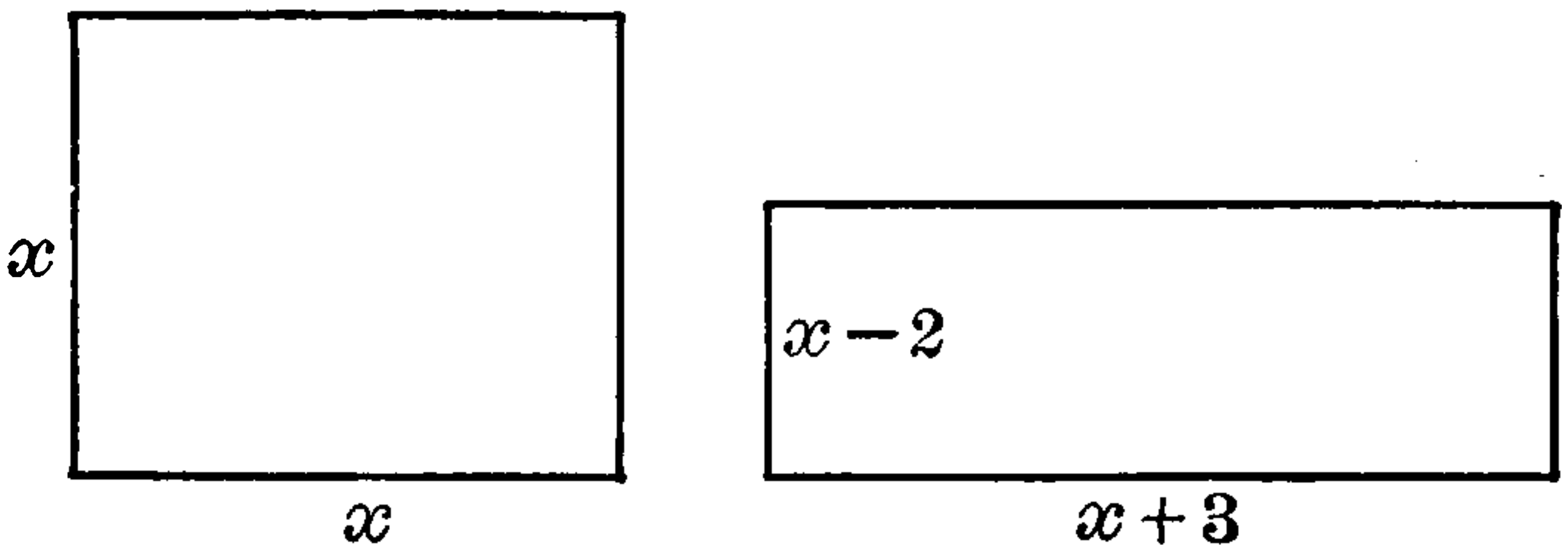
29. Find the number whose double exceeds 24 by 6.

30. The perimeter of a given rectangle is 26 feet, and the length of the rectangle exceeds the width by 5 feet. Find the dimensions of the rectangle.



31. The perimeter of a given rectangle is 18 yards, and the length exceeds the width by 3 ft. Find the dimensions. Make up and work a similar example for yourself.

32. The length of a rectangle exceeds a side of a given square by 3 inches and the width of the rectangle is 2 inches less than a side of the square. If the area of the rectangle equals the area of the square, find a side of the square.



Sug. Denote the sides of the square and rectangle as in the diagram.

Since the areas of the two figures are equal,

$$x^2 = (x + 3)(x - 2), \text{ etc.}$$

In working Exs. 33–38, draw a diagram for each example.

33. The length of a rectangle exceeds a side of a given square by 8 ft. and the width of the rectangle is 4 ft. less than a side of the square. If the area of the rectangle equals the area of the square, find a side of the square.

34. If one side of a square is increased by 4 yd., and an adjacent side by 3 yd., a rectangle is formed whose area exceeds that of the square by 47 sq. yd. Find a side of the square.

35. The perimeter of a rectangle is 120 ft., and the rectangle is twice as long as it is wide. Find its dimensions.

36. A certain rectangle is three times as long as it is wide. If 20 ft. is added to its length and 10 ft. is deducted from its width, the area is diminished by 400 sq. ft. Find the dimensions of the rectangle.

37. A rectangle is 5 ft. longer than it is wide. If its length is increased by 4 ft., and its width by 3 ft., its area is increased by 76 sq. ft. Find the dimensions of the rectangle.

38. A rectangle is 4 in. longer than it is wide. If its length is increased by 4 in., and its width diminished by 2 in., its area remains unchanged. Find the dimensions of the rectangle.

39. Make up and work an example similar to Ex. 38.

40. A tennis court is 42 ft. longer than it is wide. If a margin of 15 ft. on each end and of 10 ft. on each side is added, the area of the court is increased by 3240 sq. ft. Find the dimensions of the court.

41. The length of a football field exceeds its width by 170 ft. If a margin of 20 ft. is added on each side and end of the field, the area is increased by 21,200 sq. ft. Find the dimensions of the field.

42. A boy is three times as old as his brother. Five years hence he will be only twice as old. Find the present age of each.

43. A man is twice as old as his brother. Five years ago he was three times as old. Find the age of each at the present time.

44. How many pounds of coffee at 30¢ a pound must be mixed with 12 pounds of coffee at 20¢ a pound to make a mixture worth 24¢ a pound?

45. How many pounds of tea at 60¢ a pound must be mixed with 25 lb. of tea at 40¢ a pound, to make a mixture worth 45¢ a pound?

46. Make up and work an example similar to Ex. 45.

47. Find five consecutive numbers whose sum shall be 3 less than six times the least.

48. Find three consecutive odd numbers whose sum is 63.

49. A telegram at a 25–2 rate cost 47 cents. How many words were in the telegram?

SUG. A 25–2 rate means a cost of 25 cents for the first 10 words and 2 cents for each additional word.

50. Make up and work an example, similar to Ex. 49, concerning a telegram sent at a 40–3 rate.

51. A talk over a long distance telephone at a 50–7 rate cost 85¢. How many minutes did the talk last?

SUG. A 50–7 rate over a long distance telephone means a cost of 50 cents for the first 3 minutes and 7 cents for each additional minute.

52. A rectangle is 8 ft. longer than it is wide and the perimeter is 120 ft. Find the dimensions of the rectangle.

53. If 5 is subtracted from a certain number and the difference is subtracted from 115, the result equals three times the given number. Find the number.

54. If $\frac{1}{4}$ is added to double a certain fraction, the result is the same as if $\frac{5}{8}$ had been subtracted from three times the fraction. Find the fraction.

55. What number subtracted from 100 gives a result equal to the sum of 14 and the number?

56. Find the number which exceeds 12 by as much as three times the number exceeds 24.

57. Find five consecutive numbers such that the last is twice the first.

58. Find two consecutive integers such that the first plus 5 times the second equals 53.

59. A man is 48 years old and his son is 18. How many years ago was the father four times as old as the son? Also how many years hence will the father be twice as old as the son?

60. Find two numbers such that their difference is 20, and one is four times as large as the other.

61. The length of a single tennis court exceeds the width by 51 ft. If the width is increased by 9 ft., we have a double court, the area of which exceeds that of the single court by 702 sq. ft. Find the dimensions of each court.

62. A boy sold a certain number of newspapers on Monday, twice as many on Tuesday, on Wednesday 5 more than on Monday, and on Thursday 7 less than on Tuesday. If he sold 310 newspapers on the four days, how many did he sell on each of the days?

63. Twenty-five men agreed to pay equal amounts in raising a certain sum of money. Five of them failed to pay their subscriptions, and as a result each of the other twenty had to pay one dollar more. How much did each man subscribe originally?

64. A boy starts from a certain place and walks at the rate of 3 miles an hour. Three hours later another boy starts after the first boy and travels on a bicycle at the rate of 6 miles an hour. How many hours will it be before the second boy overtakes the first? (Draw a diagram.)

65. If the boys had traveled in opposite directions, how many hours after the second boy started would it have been before they were 81 miles apart?

66. A boy was engaged to work 50 days at 75¢ per day for the days he worked, and to forfeit 25¢ every day he was idle. On settlement he received $25.50; how many days did he work?

67. Which of the above problems belong to, or are variations of, Type I? Of Type II? III?

68. How many examples in Exercise 15 (p. 60) can you now work at sight?

CHAPTER VII

ABBREVIATED MULTIPLICATION AND DIVISION

ABBREVIATED MULTIPLICATION

74. Utility of Abbreviated Multiplication. In certain cases of multiplication, by observing the character of the expressions to be multiplied, it is possible to write out the product at once, without the labor of the actual multiplication. This is true of almost all the multiplication of binomials, and that of many trinomials, and by the use of the abbreviated methods at least three fourths of the labor of multiplication in such cases may be saved. The student should therefore master these short methods as thoroughly as the multiplication table in arithmetic.

75. I. Square of the Sum of Two Quantities.

Let $a + b$ be the sum of any two algebraic quantities.

By actual multiplication,

$$\begin{array}{lll} a + b & & \\ \underline{a + b} & & \\ a^2 + ab & & \\ \quad\; + ab & + b^2 & \\ \hline a^2 + 2ab & + b^2 & \textit{Product} \end{array}$$

Or, in brief, $(a + b)^2 = a^2 + 2ab + b^2$,

which, stated in general language, is the rule:

The square of the sum of two quantities equals the square of the first, plus twice the product of the first by the second, plus the square of the second.

Ex. 1. $(2x+3y)^2 = 4x^2 + 12xy + 9y^2$ *Product*

Ex. 2. $104^2 = (100+4)^2 = 100^2 + 8 \times 100 + 4^2$
$= 10{,}000 + 800 + 16 = 10816$ *Ans.*

76. II. Square of the Difference of Two Quantities.

By actual multiplication,

$$\begin{array}{rrr} a & -\,b & \\ a & -\,b & \\ \hline a^2 & -\,ab & \\ & -\,ab & +\,b^2 \\ \hline a^2 & -\,2ab & +\,b^2 \end{array} \quad \textit{Product}$$

Or, in brief, $(a-b)^2 = a^2 - 2ab + b^2$,

which, stated in general language, is the rule:

The square of the difference of two quantities equals the square of the first, minus twice the product of the first by the second, plus the square of the second.

Ex. 1. $(2x-3y)^2 = 4x^2 - 12xy + 9y^2$ *Product*

Ex. 2. $[(x+2y)-5]^2 = (x+2y)^2 - 10(x+2y) + 25$
$= x^2 + 4xy + 4y^2 - 10x - 20y + 25$ *Product*

To check the work of Ex. 2, let $x = 2$, $y = 1$.
Then $[(x^2+2y)-5]^2 = (2+2-5)^2 = (-1)^2 = 1$
Also
$x^2 + 4xy + 4y^2 - 10x - 20y + 25 = 4 + 8 + 4 - 20 - 20 + 25 = 1$.

EXERCISE 26

Write by inspection the value of each of the following and check each result:

1. $(n+y)^2$
2. $(c-x)^2$
3. $(2x-y)^2$
4. $(3x-2y)^2$
5. $(5x+1)^2$
6. $(x^2+1)^2$
7. $(x-y^2)^2$
8. $(1-7y^3)^2$

9. $(3x^4 + 5x^3)^2$

10. $(6x^2y - 11y^2z^3)^2$

11. $(5x^n - 3y^nz^m)^2$

12. $(4x^3y^5z^{2n} + 9y^{3n})^2$

13. $(\frac{1}{2}x^3 + \frac{2}{3}y)^2$

14. $(\frac{3}{4}ab - \frac{5}{6}x^2)^2$

15. $(.2x + .3y)^2$

16. $(.3a + .04b^2)^2$

17. $(.02x - .03y)^2$

18. $(1.5m - .02)^2$

19. $[(a + b) + 4]^2$

20. $[(a + b) - 3]^2$

21. $[(a + b) + c]^2$

22. $[(2a - x) + 3y]^2$

23. $[3 + (a + b)]^2$

24. $[5a - (x + y)]^2$

25. $[2a^2 - (b - 2c)]^2$

26. $[(x + y) - (a + b)]^2$

27. Find the value of 998 by multiplying 998 by itself. This product might also have been obtained in the following way:

$$\begin{aligned} 998^2 = (1000 - 2)^2[&= 1000^2 - 2 \times 2 \times 1000 + 2^2] \\ &= 1{,}000{,}000 - 4000 + 4 \\ &= 996{,}004 \end{aligned}$$

After practice the part of the work in the brackets may be omitted. Compare the amount of work in the two processes of finding the value of 998^2.

By the short method obtain the value of:

28. 999^2

29. 997^2

30. 9998^2

31. 51^2

32. 1003^2

33. 97^2

34. 996^2

35. 9997^2

36. $(99.2)^2$

37. Make up and work an example similar to Ex. 19. To Ex. 29. Ex. 36.

38. How many of the examples in this Exercise can you answer orally?

77. III. Product of the Sum and Difference of Two Quantities.

By actual multiplication,

$$\begin{array}{lll} a + b & & \\ \underline{a - b} & & \\ a^2 + ab & & \\ \underline{\quad - ab - b^2} & & \\ a^2 \qquad - b^2 & \textit{Product} \end{array}$$

Or, in brief, $(a + b)(a - b) = a^2 - b^2$,

which, stated in general language, is the rule:

The product of the sum and difference of two quantities equals the square of the first minus the square of the second.

Ex. 1. $(2x + 3y)(2x - 3y) = 4x^2 - 9y^2$ *Product*

Ex. 2. Multiply $x + (a + b)$ by $x - (a + b)$.

We have

$$\begin{aligned} [x + (a + b)][x - (a + b)] &= x^2 - (a + b)^2, \text{ by III.} \\ &= x^2 - (a^2 + 2ab + b^2), \text{ by I.} \\ &= x^2 - a^2 - 2ab - b^2 \ \textit{Product} \end{aligned}$$

Let the pupil check the work.

It is frequently necessary to re-group the terms of trinomials in order that the multiplication may be performed by the above method.

Ex. 3. Multiply $x + y - z$ by $x - y + z$.

$$\begin{aligned} (x + y - z)(x - y + z) &= [x + (y - z)][x - (y - z)] \\ &= x^2 - (y - z)^2, \text{ by III.} \\ &= x^2 - (y^2 - 2yz + z^2), \text{ by II.} \\ &= x^2 - y^2 + 2yz - z^2 \quad \textit{Product} \end{aligned}$$

Let the pupil check the work.

EXERCISE 27

Write by inspection the value of each of the following products, and check the work for each result:

1. $(x + z)(x - z)$	3. $(3x - y)(3x + y)$
2. $(y - 3)(y + 3)$	4. $(7x + 4y)(7x - 4y)$

5. $(x^2 - 2)(x^2 + 2)$
6. $(ax^2 - b^2y)(ax^2 + b^2y)$
7. $(1 - 11x^3)(1 + 11x^3)$
8. $(2x^n + 5y^m)(2x^n - 5y^m)$
9. $(\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a - \frac{1}{3}b)$
10. $(2\frac{1}{4}x - \frac{1}{3}y)(2\frac{1}{4}x + \frac{1}{3}y)$
11. $(.2x + .3y)(.2x - .3y)$
12. $(.05a^2 - .3b^3)(.05a^2 + .3b^3)$
13. $(\frac{3}{5}x + .7y)(\frac{3}{5}x - .7y)$
14. $(a^{n+1} + \frac{1}{2}b^{n-1})(a^{n+1} - \frac{1}{2}b^{n-1})$
15. $[(a + b) + 3][(a + b) - 3]$
16. $[(x + y) + a][(x + y) - a]$
17. $[(2x - 1) + y][(2x - 1) - y]$
18. $[4 + (x + 1)][4 - (x + 1)]$
19. $[2x + (3y - 5)][2x - (3y - 5)]$
20. $(a + b + 3)(a + b - 3)$
21. $(x + y + a)(x + y - a)$
22. $(4 + x + 1)(4 - x - 1)$
23. $(2x + 3y - 5)(2x - 3y + 5)$
24. $(4 + x + y)(4 - x - y)$
25. $(x^2 + 3x + 2)(x^2 + 3x - 2)$
26. $(a + b + 3x)(a + b - 3x)$
27. $(a + b - 3x)(a - b + 3x)$
28. $(x^2 - xy + y^2)(x^2 + xy + y^2)$
29. $(a^2 + a + 1)(a^2 - a + 1)$
30. $(2x^2 - 3x - 5)(2x^2 + 3x - 5)$
31. $(2x^2 + 5xy - y^2)(2x^2 - 5xy - y^2)$
32. $(x^2 + xy - y^2)(x^2 - xy - y^2)$
33. $[(a + b) - (c - 1)][(a + b) + (c - 1)]$
34. $[(x^2 + y^2) + (x^2y^2 + 1)][(x^2 + y^2) - (x^2y^2 + 1)]$
35. $(x + y + z + 1)(x + y - z - 1)$

36. Work Ex. 16 in full (see Art. 54, p. 65). How much of this labor is saved by the short method of multiplication?

37. Make up and work an example similar to Ex. 36.

38. Multiply 93 by 87. This product may also be obtained thus:

$$\begin{aligned} 93 \times 87 &= (90 + 3)(90 - 3) \\ &= 8100 - 9 = 8091 \end{aligned}$$

Compare the amount of work in the two processes.

39. Make up and work an example similar to Ex. 38.

Find the value of each of the following in the short way:

40. 92×88

41. 103×97

42. 105×95

43. 1005×995

44. $103^2 - 97^2$

45. $(17.31)^2 - (2.69)^2$

Find in the shortest way:

46. The area of a rectangle 102 ft. long and 98 ft. wide.

47. The cost of 32 doz. eggs at 28¢ per dozen.

48. The cost of 67 yd. of cloth at 73¢ a yard.

49. Make up and work two examples similar to Exs. 47–48.

50. Work Ex. 40 in full. How much of this labor is saved by using the short method of multiplication?

Write at sight the product for each of the following miscellaneous examples:

51. $(x + 2a)^2$

52. $(x + 2a)(x - 2a)$

53. $(x - 2a)^2$

54. $(3x - 1)(3x + 1)$

55. $(3x - 1)^2$

56. $(3a^2 - 2b^3)^2$

57. $(3a^2 - 2b^3)(3a^2 + 2b^3)$

58. $[(x + 2y) + 5]^2$

59. $(x + 2y + 5)(x + 2y - 5)$

60. $(.3x + .5y)(.3x - .5y)$

61. 998^2

62. 998×1002

63. 97^2

64. 97×103

65. Make up and work an example in each principal form of abbreviated multiplication studied thus far.

66. How many of the examples in this Exercise can you answer orally?

78. IV. Square of any Polynomial.

By actual multiplication,

$$\begin{array}{llllll}
a & +b & +c & & & \\
a & +b & +c & & & \\
\hline
a^2 & +ab & +ac & & & \\
 & +ab & & +b^2 & +bc & \\
 & & +ac & & +bc & +c^2 \\
\hline
a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2 \quad \textit{Product}
\end{array}$$

Or, in brief, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

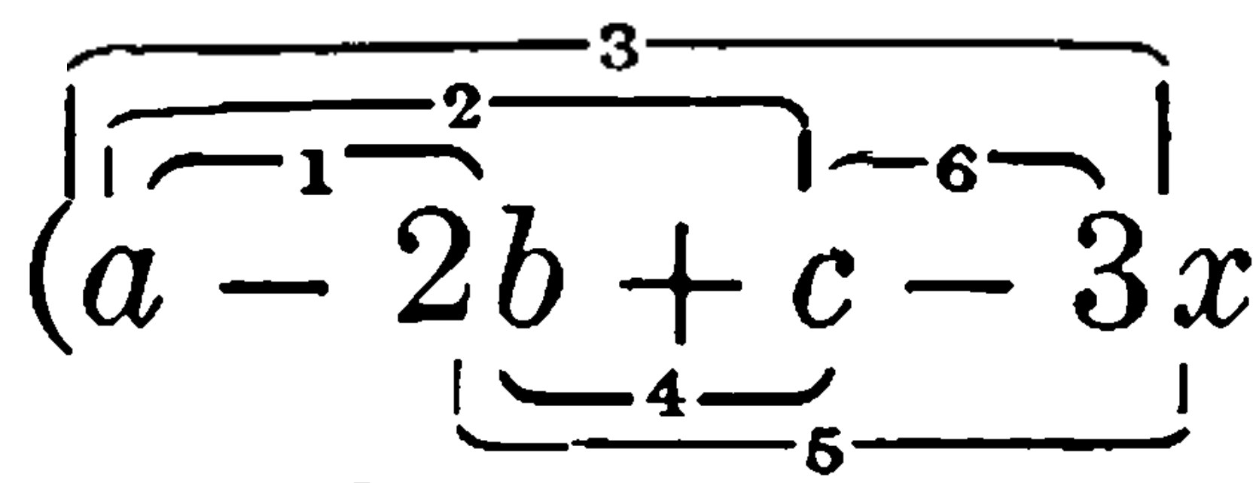
In like manner we obtain

$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

Or, in general,

The square of any polynomial equals the sum of the squares of the terms plus twice the product of each term by each term which follows it.

It is often useful to indicate the order in which the products of the terms are taken as shown in the following diagram. (If the curved lines joining the terms are drawn as each product is taken, the numbers on these lines may be omitted.)



Ex. $(a - 2b + c - 3x)^2 = a^2 + 4b^2 + c^2 + 9x^2 - 4ab + 2ac - 6ax - 4bc + 12bx - 6cx.$

Let the pupil check the work.

EXERCISE 28

Find in the shortest way the value of the following and check each result:

1. $(2x + y + 1)^2$
2. $(x - 2y + 2z)^2$
3. $(3x - 2y - 5)^2$
4. $(2a - b + 3c)^2$
5. $(x - 2y - 3z)^2$
6. $(4x + 3y - 1)^2$
7. $(x^2 - x + 1)^2$
8. $(2a^2 + 5a - 3)^2$
9. $(x - y + z - 1)^2$
10. $(2x + 3y - 4z - 5)^2$
11. $(3x^3 - 4x^2 + x - 2)^2$
12. $(\frac{1}{2}x^2 - \frac{2}{3}x + 5)^2$
13. $(\frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{3}{4}x + 6)^2$
14. $(.2a + .3b - .5c)^2$

15. Expand $(2a - 3b + c - 4d)^2$ by multiplying in full. Now obtain the same result by the method of Art. 78, p. 115. About how much of the work of multiplication is saved by the latter method?

16. Make up and work an example similar to Ex. 15.

Write at sight the product for each of the following miscellaneous examples:

17. $(2a - 3b)^2$
18. $(2a + 3b)(2a - 3b)$
19. $(2 + 3b)^2$
20. $(3x^2 - y^3)(3x^2 + y^3)$
21. $(3x^2 + y^3)^2$
22. $(3x^2 - 4y^3)^2$
23. $(x + 2y - 3)(x + 2y + 3)$
24. $(x + 2y - 3)^2$
25. $(4x + \frac{1}{2}a - \frac{2}{3})^2$
26. $(4x + \frac{1}{2}a - \frac{2}{3})(4x + \frac{1}{2}a + \frac{2}{3})$
27. $(x^{n+1} - 3x^{n-1}y)^2$
28. $(x^{n+1} - 3x^{n-1}y)(x^{n+1} + 3x^{n-1}y)$
29. $(.02x^2 - .3x + .5)^2$

30. Make up and work an example in each principal form of abbreviated multiplication studied thus far.

79. V. Product of Two Binomials of the Form $x + a$, $x + b$.

By actual multiplication,

$$\begin{array}{l} x + 5 \\ x + 3 \\ \hline x^2 + 5x \\ \qquad + 3x + 15 \\ \hline x^2 + 8x + 15 \end{array} \qquad \begin{array}{l} x - 5y \\ x + 3y \\ \hline x^2 - 5xy \\ \qquad + 3xy - 15y^2 \\ \hline x^2 - 2xy - 15y^2 \end{array} \qquad \begin{array}{l} x + a \\ x + b \\ \hline x^2 + ax \\ \qquad + bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

By comparing each pair of binomials with their product, we observe the following relation:

The product of two binomials of the form $x + a$ and $x + b$ consists of three terms:

The first term is the square of the first term of the binomials;

The last term is the product of the second terms of the binomials;

The middle term consists of the first term of the binomials with a coefficient equal to the algebraic sum of the second terms of the binomials.

Ex. 1. Multiply $x - 8$ by $x + 7$.

$$-8 + 7 = -1. \qquad -8 \times 7 = -56.$$
$$\therefore (x - 8)(x - 7) = x^2 - x - 56 \text{ Product}$$

Ex. 2. Multiply $(x - 6a)(x - 5a)$.

$$(-6a) + (-5a) = -11a. \qquad (-6a) \times (-5a) = +30a^2.$$
$$\therefore (x - 6a)(x - 5a) = x^2 - 11ax + 30a^2 \text{ Product}$$

Ex. 3. Multiply $x + y + 6$ by $x + y - 2$.

$$\begin{aligned} (x + y + 6)(x + y - 2) &= [(x + y) + 6][(x + y) - 2] \\ &= (x + y)^2 + 4(x + y) - 12 \text{ Product} \end{aligned}$$

EXERCISE 29

Write the product for each of the following and check each result:

1. $(x + 2)(x + 5)$

2. $(x - 5)(x - 3)$

3. $(x - 7)(x + 4)$

4. $(x - 4)(x + 8)$

5. $(x + 1)(x - 7)$

6. $(x^2 - 2)(x^2 - 3)$

7. $(x^2 + 3)(x^2 + 1)$

8. $(a + 3x)(a - 10x)$

9. $(x - 7y)(x + y)$

10. $(x + .2)(x + .5)$

11. $(x + \frac{1}{2})(x + \frac{1}{3})$

12. $(x + .02)(x + 5)$

13. $(a + .02)(a + .5)$

14. $(x - \frac{1}{2})(x + \frac{1}{3})$

15. $(a + \frac{2}{3})(a - \frac{1}{2})$

16. $(ab + x)(ab + 3x)$

17. $(ab + x)(ab - 3x)$

18. $(xy - 7z^2)(xy + 3z^2)$

19. $(x^n + 5)(x^n - 5)$

20. $[(x + y) + 3][(x + y) + 5]$

21. $[(x + y) - 3][(x + y) + 5]$

22. $(x + y - 3)(x + y + 5)$

23. $(a + 2b + 5)(a + 2b + 3)$

24. $(2x + 3y + 3a)(2x + 3y - 5a)$

25. $(x + a + b)(x - a - b)$

26. $(2x + a + 3b)(2x - a - 3b)$

27. $(2x + a - 3b)(2x - a + 3b)$

28. Find the product of $x + y + 6$ and $x + y - 3$ by multiplying in full. Then find the same product by the method of Art. 79. About how much of the work of multiplication is saved by use of the latter method?

29. Make up and work an example similar to Ex. 27.

30. A building lot is 167 ft. wide and 213 ft. deep. If the width and depth of the lot are each increased by 1 foot, find the increase in area without multiplying 167 by 213.

Write at sight the product for each of the following miscellaneous examples:

31. $(x + 5)(x - 5)$
32. $(x + 5)^2$
33. $(x - 5)^2$
34. $(x + 5)(x - 3)$
35. $(x + 5)(x - 3a)$
36. $(x - 3a)(x + 3a)$
37. $(x - 3a)(x + 5a)$
38. $(x - 5a)^2$
39. $(x + y + 5a)(x + y - 5a)$
40. $(x + y + 5a)^2$
41. $(x + y + 5a)(x + y + 3a)$
42. $(x + y + 5a)(x + y - 3a)$
43. $(x^2 + \frac{1}{2}x + 3)^2$
44. $(x^2 + \frac{1}{2}x + 3)(x^2 + \frac{1}{2}x - 3)$
45. $(x^2 + \frac{1}{2}x + 3)(x^2 + \frac{1}{2}x - 5)$
46. $(a^2 + ax + x^2)(a^2 - ax + x^2)$

47. Make up and work an example in each principal form of abbreviated multiplication studied thus far.

48. How many of Exs. 31–45 can you answer orally?

80. VI. Product of Two Binomials whose Corresponding Terms are Similar.

By actual multiplication,

$$\begin{array}{lll}
2a - 3b & & \\
\underline{4a + 5b} & & \\
8a^2 - 12ab & & \\
 + 10ab & - 15b^2 & \\
\hline
8a^2 - 2ab & - 15b^2 & \textit{Product}
\end{array}$$

We see that the middle term of this product may be obtained directly from the two binomials by taking the algebraic sum of the cross products of their terms. Thus,

$$(+2a)(+5b) + (-3b)(+4a) = 10ab - 12ab = -2ab.$$

Hence, in general,

The product of any two binomials of the given form consists of three terms:

The first term is the product of the first terms of the binomials;

The third term is the product of the second terms of the binomials;

The middle term is formed by taking the algebraic sum of the cross products of the terms of the binomials.

Ex. Multiply $10x + 7y$ by $8x - 11y$.

To show the method of obtaining the middle term of the product, we write the given expression in the form

$$(10x + 7y)(8x - 11y)$$

Hence,

$$(10x)(-11y) + (7y)(8x) = -110xy + 56xy = -54xy$$
$$\therefore (10x + 7y)(8x - 11y) = 80x^2 - 54xy - 77y^2 \text{ Product}$$

EXERCISE 30

Write at sight the product of each of the following and check each result:

1. $(2x + 3)(x + 4)$
2. $(2x - 3)(x - 4)$
3. $(2x + 3)(x - 4)$
4. $(2x - 3)(x + 4)$
5. $(3a + 5)(2a + 3)$
6. $(3a - 5)(2a + 3)$
7. $(5x - 1)(x + 7)$
8. $(x + 3y)(3x - 8y)$
9. $(3a^2 + b)(4a^2 - 5b)$
10. $(x + \frac{1}{2})(\frac{3}{4}x + \frac{1}{2})$
11. $(a + .2b)(2a - .3b)$
12. $(\frac{1}{2}x + \frac{3}{2}a)(\frac{3}{2}x - \frac{1}{2}a)$

13. How many examples in Exercise 9 (p. 41) can you now work at sight?

EXERCISE 31

REVIEW

Write at sight the value of each of the following and check each result:

1. $(2x+3)^2$

2. $(2x-3)^2$

3. $(2x+3)(2x-3)$

4. $(x+3)(x-5)$

5. $(2x+3)(3x-5)$

6. $(x+3y)(x+2y)$

7. $(2x+3y)^2$

8. $(2x+3y)(3x-4y)$

9. $(2x-3y)^2$

10. $(2x+3y)(2x-3y)$

11. $(5a-3x)(4a+5x)$

12. $(7x+3y^2)^2$

13. $(5a^2+3y^3)(5a^2-3y^3)$

14. $(a^2+3x)(a^2-5x)$

15. $(2a+3x+5)^2$

16. $(2a+3x+5)(2a+3x-5)$

17. $(2a+3x+5)(2a+3x-3)$

18. $(1-2x-3x^2+x^3)^2$

19. $(a+b+x+y)(a+b-x-y)$

20. $(a^2+ax+x^2)(a^2-ax+x^2)$

21. $(x^{a+b}-x^{a-b})^2$

22. $(4a^2+2a+1)(4a^2-2a+1)$

23. $(3ax^n-2a^{n-1}x)^2$

24. $(x^n+2xy^{n-1})^2$

25. $(1-a)^2$

26. $(a-1)^2$

27. $(-2x+3y)^2$

28. $(-2x-3y)^2$

29. $(-a-b)(-a+b)$

30. $(-x+3)(-x-3)$

31. Why is it that the result of expanding $(-2x-3y)^2$ is the same as that of expanding $(2x+3y)^2$?

32. Give two expressions similar to those in Ex. 31 for which the product is the same.

33. Why is $(a-b)^2$ equal to $(b-a)^2$? Make up two expressions similar to these.

34 Make up and work an example in each principal form of abbreviated multiplication studied thus far.

Simplify, using the methods of abbreviated multiplication as far as possible:

35. $(a + 2b)^2 + (a - 2b)^2$

36. $(a + 2b)^2 - (a - 2b)^2$

37. $(2x - 1)^2 + (1 - 2x)^2$

38. $(2x - 1)^2 - (2x + 1)^2$

39. $(3a - 1)^2 + (2 - 3a)(2 + 3a)$

40. $(2x - 7y)(2x + 7y) - 4(x - 2y)^2 + 13y(5y - x)$

41. $(3x^2 + 5)^2 + x^2(10 - 3x)(10 + 3x) - (5 + 13x^2)^2$

42. $(a - c + 1)(a + c - 1) - (a - 1)^2 + 2(c - 1)^2$

43. $(x + y - xy)(x - y - xy) + x^2y - (x - y^2)(x + y^2)$

44. Show that $a^2 = (a + b)(a - b) + b^2$.

45. By use of the relation proved in Ex. 44, obtain the value of $(7\frac{1}{2})^2$ in a short way.

SUG. We have
$$\begin{aligned}(7\tfrac{1}{2})^2 &= (7\tfrac{1}{2} + \tfrac{1}{2})(7\tfrac{1}{2} - \tfrac{1}{2}) + (\tfrac{1}{2})^2\\ &= 8 \times 7 + \tfrac{1}{4} = 56\tfrac{1}{4}\ \textit{Ans.}\end{aligned}$$

Using the method of Ex. 45 find the value of:

46. $(8\frac{1}{2})^2$

47. $(19\frac{1}{2})^2$

48. $(199\frac{1}{2})^2$

49. $(15\frac{1}{2})^2$

50. $(49\frac{1}{2})^2$

51. $(99\frac{1}{2})^2$

52. $(7.5)^2$

53. $(19.5)^2$

54. $(99.5)^2$

55. $(75)^2$

56. $(195)^2$

57. $(995)^2$

58. $(9.7)^2$ (Use $(9.7)^2 = 10 \times 9.4 + .3^2$)

59. $(9.8)^2$

60. $(9.6)^2$

61. $(4.8)^2$

62. 98^2

63. Find the value of $(a + b)^3$ by multiplication. Examine the result obtained. Make a rule for obtaining similar products in a short way. Treat $(a - b)^3$ in the same way.

64. By use of the rule obtained in Ex. 63, write out by inspection the value of $(x + y)^3$.

65. Also of $(a - x)^3$.

66. Of $(b + y)^3$.

Solve the following equations, using methods of abbreviated multiplication wherever possible:

67. $(2x - 1)^2 - 4x^2 = 19$

68. $(2x + 1)^2 - (2x - 1)^2 = 16$

Compute in the shortest way:

69. The area of a field 103 rd. long and 97 rd. wide.

70. The area of a square field each side of which is 98 rd.

71. The cost of 62 yd. of cloth at 58¢ per yard.

72. The cost of 85 A. of land at $95 per A.

73. How many of the examples in this exercise can you work at sight?

ABBREVIATED DIVISION

81. Utility of Abbreviated Division. In certain cases much of the labor of division may be saved by the use of mechanical rules. We discover these rules by performing the division operation in a typical case, noting the relation between the quantities divided and the quotient, and formulating this relation into a rule.

82. I. Division of the Difference of Two Squares.

Either by actual division, or by inverting the relation of Art. 77, p. 112, we obtain

$$\frac{a^2 - b^2}{a + b} = a - b \qquad \text{and} \qquad \frac{a^2 - b^2}{a - b} = a + b.$$

Hence, in general language,

The difference of the squares of two quantities is divisible by the sum of the quantities, and also by the difference of the quantities, the quotients in the respective cases being the difference of the quantities and the sum of the quantities.

Ex. 1. $\frac{4x^2 - 9y^2}{2x - 3y} = 2x + 3y$ *Quotient*

Ex. 2. $\frac{x^2 - (a + b)^2}{x + (a + b)} = x - (a + b)$ *Quotient*

Let the pupil check the work in these examples.

EXERCISE 32

Write at sight the quotient for each of the following, and check each result:

1. $\dfrac{a^2 - x^2}{a - x}$

2. $\dfrac{9 - 4x^2}{3 - 2x}$

3. $\dfrac{x^2 - 81y^2}{x + 9y}$

4. $\dfrac{25x^2 - 36y^4}{5x - 6y^2}$

5. $\dfrac{16x^4 - 49y^4}{4x^2 + 7y^2}$

6. $\dfrac{25x^{10} - y^{12}}{5x^5 - y^6}$

7. $\dfrac{a^2b^4 - 36c^6d^8}{ab^2 + 6c^3d^4}$

8. $\dfrac{\frac{1}{4}x^2 - \frac{4}{9}y^2}{\frac{1}{2}x - \frac{2}{3}y}$

9. $\dfrac{\frac{9}{16}a^4 - \frac{1}{9}x^6}{\frac{3}{4}a^2 - \frac{1}{3}x^3}$

10 $\dfrac{.25a^2 - .16b^2}{.5a - .4b}$

11. $\dfrac{.04x^2 - .09y^2}{.2x + .3y}$

12. $\dfrac{x^2 - .25b^4}{x - .5b^2}$

13. Divide $a^2 + 2ab + b^2 - 4x^2$ by $a + b - 2x$ by long division. Write the result of dividing $(a + b)^2 - 4x^2$ by $a + b - 2x$ by the method of Art. 82. Estimate how much less the labor of the second process is than that of the first.

14. Make up and solve an example similar to Ex. 13.

Obtain in the shortest way the quotient for each of the following:

15. $\dfrac{(x + 1)^2 - a^2}{x + 1 + a}$

16. $\dfrac{a^2 - (b - 2c)^2}{a - (b - 2c)}$

17. $\dfrac{4x^4 - (y^2 + 1)^2}{2x^2 + (y^2 + 1)}$

18. $\dfrac{(a - b)^2 - (c - 1)^2}{(a - b) + (c - 1)}$

19. $\dfrac{1 - (a + b - c)^2}{1 + (a + b - c)}$

20. $\dfrac{(2a + 3b)^2 - (5x - 4y)^2}{(2a + 3b) - (5x - 4y)}$

Write a divisor and quotient for each of the following:

21. $\dfrac{a^2 - 4x^2}{\quad} =$ **24.** $\dfrac{9a^2 - (x + y)^2}{\quad} =$

22. $\dfrac{a^4 - 4x^6}{\quad} =$ **25.** $\dfrac{(a + x)^2 - (b + y)^2}{\quad} =$

23. $\dfrac{x^6 - y^6}{\quad} =$ **26.** $\dfrac{4(a + b)^2 - 9x^2}{\quad} =$

Find two factors for each of the following:

27. 2500 − 16 **29.** 2491 **31.** 99.19

28. 2484 **30.** 9919 **32.** 6319

33. Divide $a^2 - b^2$ by $a - b$. Divide $(a - b)^2$ by $(a - b)$.

34. Find the difference in value between $(x + y)^2$ and $x^2 + y^2$, when $x = 2$ and $y = 3$.

83. II and III. Division of Sum or Difference of Two Cubes.

By actual division we can obtain,

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2, \quad \text{and} \quad \frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

Hence, in general language,

The **sum** *of the cubes of two quantities is divisible by the sum of the quantities, and the quotient is the square of the first quantity, minus the product of the two quantities, plus the square of the second quantity;* also

The **difference** *of the cubes of two quantities is divisible by the difference of the quantities, and the quotient is the square of the first quantity, plus the product of the two quantities, plus the square of the second.*

Ex. 1. $$\begin{aligned}\frac{8x^3-27y^3}{2x-3y}&=\frac{(2x)^3-(3y)^3}{2x-3y}\\&=(2x)^2+(2x)(3y)+(3y)^2\\&=4x^2+6xy+9y^2 \textit{ Quotient}\end{aligned}$$

Ex. 2. $$\begin{aligned}\frac{(a-b)^3+27}{(a-b)+3}&=(a-b)^2-3(a-b)+9\\&=a^2-2ab+b^2-3a+3b+9 \textit{ Quotient}\end{aligned}$$

Let the pupil check the work in these examples.

EXERCISE 33

Write at sight the quotient for each of the following and check each result:

1. $\dfrac{a^3+8}{a+2}$

2. $\dfrac{x^3-1}{x-1}$

3. $\dfrac{27x^3-64}{3x-4}$

4. $\dfrac{1+8x^6}{1+2x^2}$

5. $\dfrac{125-x^9}{5-x^3}$

6. $\dfrac{27a^6+y^{12}}{3a^2+y^4}$

7. $\dfrac{x^6+y^6}{x^2+y^2}$

8. $\dfrac{.008x^3-y^3}{.2x-y}$

9. $\dfrac{\frac{1}{8}x^3-\frac{1}{27}y^3}{\frac{1}{2}x-\frac{1}{3}y}$

10. $\dfrac{\frac{8}{27}a^3+\frac{1}{64}b^6}{\frac{2}{3}a+\frac{1}{4}b^2}$

11. Divide $8a^3+27b^3$ by $2a+3b$ by the method of long division. Now write out the same quotient by the method of Art. 83. Estimate how much of the labor of division is saved by using the second method of obtaining the quotient.

12. Make up and work an example similar to Ex. 11.

13. Treat $(a+b)^3-8x^3$ divided by $(a+b)-2x$ as in Ex. 11.

Obtain in the shortest way the quotient for each of the following:

14. $\dfrac{c^3 + (1 - x)^3}{c + (1 - x)}$

15. $\dfrac{8 - (x + y)^3}{2 - (x + y)}$

16. $\dfrac{27x^6 + 125y^9}{3x^2 + 5y^3}$

17. $\dfrac{(a - 1)^3 - x^6}{(a - 1) - x^2}$

18. $\dfrac{8x^3 + (x^2 - 1)^3}{2x + x^2 - 1}$

19. $\dfrac{8(x - y)^3 - z^3}{2(x - y) - z}$

Write the binomial divisor and the quotient for

20. $\dfrac{8a^3 - x^3}{} =$

21. $\dfrac{8a^3 - 27x^3}{} =$

22. $\dfrac{x^3 + 8b^3}{} =$

23. $\dfrac{1 - 64x^3}{} =$

24. $\dfrac{8a^3 + 1}{} =$.

25. $\dfrac{x^6 + y^6}{} =$

26. $\dfrac{x^{12} + y^{12}}{} =$

27. $\dfrac{8x^6 - (a + b)^3}{} =$

28. $\dfrac{(a + b)^3 + (x + y)^3}{} =$

Find a factor of each of the following:

29. 27,000 − 64

30. 26,936

31. 992

32. 973

33. 8027

34. 7936

35. Divide $a^3 - b^3$ by $a - b$. Also divide $(a - b)^3$ by $a - b$.

36. Find the difference in value between $x^3 + y^3$ and $(x + y)^3$ when $x = 2$ and $y = 3$.

Write a binomial divisor and the corresponding quotient for each of the following miscellaneous examples:

37. $\dfrac{a^2 - 4b^2}{} =$

38. $\dfrac{a^3 - 8b^3}{} =$

39. $\dfrac{a^3 + 8b^3}{} =$

40. $\dfrac{27x^3 - 1}{} =$

41. $\dfrac{9x^2 - 1}{} =$

42. $\dfrac{16a^4 - 9}{} =$

43. $\dfrac{8x^3 - 27y^3}{} =$

44. $\dfrac{x^3 + 1}{} =$

45. $\dfrac{x^2 - 1}{} =$

46. $\dfrac{1 - 27x^3}{} =$

47. $\dfrac{a^2 - 4(x + y)^2}{} =$

48. $\dfrac{a^3 - 8(x + y)^3}{} =$

49. $\dfrac{8(x + y)^3 + a^3}{} =$

50. $\dfrac{a^4 - 9(x - y)^2}{} =$

51. $\dfrac{27a^3 - (x - y)^3}{} =$

52. $\dfrac{(x + y)^2 - (x - y)^2}{} =$

53. $\dfrac{(x + 1)^3 + (x - 1)^3}{} =$

54. How many of the examples in this Exercise can you work at sight?

55. How many examples in Exercise 1 (p. 8) can you now work at sight?

84. IV, V, and VI. Division of Sum or Difference of any Two Like Powers.

By actual division we can obtain,

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3 \textit{ Quotient}$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3 \textit{ Quotient}$$

$a^4 + b^4$ is not divisible by either $a + b$ or $a - b$. But

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4 \textit{ Quotient}$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4 \textit{ Quotient}$$

Hence,

The **difference** of two like **even** powers of two quantities is divisible by the **sum** of the quantities, and also by their **difference**;

The **sum** *of two like* **odd** *powers of two quantities is divisible by the* **sum** *of the quantities;*

The **difference** *of two like* **odd** *powers of two quantities is divisible by the* **difference** *of the quantities.*

For the quotient in all these cases —

(1) The number of terms in a quotient equals the degree of the powers whose sum or difference is divided;

(2) The terms of each quotient are homogeneous (since the exponent of a decreases by 1 in each term, and that of b increases by 1 in each term).

(3) *If the divisor is a difference, the signs of the quotient are all plus; if the divisor is a sum, the signs of the quotient are alternately plus and minus.*

In the above statements the parts *in italics* should be committed to memory.

The last statement forms a general rule for signs of a quotient in all the cases of abbreviated division, including I–VI.

Ex. 1. $$\frac{32x^5+y^5}{2x+y}=\frac{(2x)^5+y^5}{2x+y}$$
$$=(2x)^4-(2x)^3y+(2x)^2y^2-(2x)y^3+y^4$$
$$=16x^4-8x^3y+4x^2y^2-2xy^3+y^4 \textit{ Quotient.}$$

Ex. 2. $$\frac{a^{10}+x^{10}}{a^2+x^2}=\frac{(a^2)^5+(x^2)^5}{a^2+x^2}$$
$$=(a^2)^4-(a^2)^3(x^2)+(a^2)^2(x^2)^2-(a^2)(x^2)^3+(x^2)^4$$
$$=a^8-a^6x^2+a^4x^4-a^2x^6+x^8$$

EXERCISE 34

Write at sight the quotient for each of the following and check each result:

1. $\frac{a^5 + x^5}{a + x}$

2. $\frac{a^5 - x^5}{a - x}$

3. $\frac{b^7 + y^7}{b + y}$

4. $\frac{b^7 - y^7}{b - y}$

5. $\frac{a^5 + 32}{a + 2}$

6. $\frac{a^7 - 128}{a - 2}$

7. $\frac{x^5 - 1}{x - 1}$

8. $\frac{x^5 + 1}{x + 1}$

9. $\frac{32x^5 - y^5}{2x - y}$

10. $\frac{a^{11} + x^{11}}{a + x}$

11. $\frac{x^{10} - y^{15}}{x^2 - y^3}$

12. $\frac{243 - a^5}{3 - a}$

13. Divide $32a^5 + x^5$ by $2a + x$ by the method of long division. Now write the same quotient by the method of Art. 84, p. 128. Estimate how much of the labor of division is saved by using the second method.

14. Make up and work an example similar to Ex. 13.

Write a binomial divisor and the corresponding quotients for each of the following:

15. $\frac{a^5 + y^5}{\qquad} =$

16. $\frac{a^5 - y^5}{\qquad} =$

17. $\frac{b^7 + x^7}{\qquad} =$

18. $\frac{b^7 - x^7}{\qquad} =$

19. $\frac{x^5 - 32}{\qquad} =$

20. $\frac{a^7 + 128}{\qquad} =$

21. $\frac{1 - 32x^5}{\qquad} =$

22. $\frac{y^5 + 1}{\qquad} =$

Obtain a factor of each of the following:

23. 100,001

24. 100,243

25. 100,032

26. 99,757

27. Divide $a^5 + b^5$ by $a + b$. Also divide $(a + b)^5$ by $a + b$.

28. Find the difference in value between $x^5 - y^5$ and $(x - y)^5$, when $x = 3$ and $y = 2$.

EXERCISE 35

Review

Write at sight the quotient for each of the following:

1. $\dfrac{b^2 - x^2}{b - x}$

2. $\dfrac{b^3 - x^3}{b - x}$

3. $\dfrac{b^5 - x^5}{b - x}$

4. $\dfrac{b^5 + x^5}{b + x}$

5. $\dfrac{b^3 + x^3}{b + x}$

6. $\dfrac{b^2 - 4x^2}{b + 2x}$

7. $\dfrac{b^3 - 8x^3}{b - 2x}$

8. $\dfrac{b^5 - 32x^5}{b - 2x}$

9. $\dfrac{b^3 + 8x^3}{b + 2x}$

10. $\dfrac{x^2 - 4(a + b)^2}{x - 2(a + b)}$

11. $\dfrac{x^3 - 8(a + b)^3}{x - 2(a + b)}$

12. $\dfrac{8x^3 - a^3}{2x - a}$

13. $\dfrac{27a^6 - 8(x + y)^3}{3a^2 - 2(x + y)}$

14. $\dfrac{x^{12} + y^9}{x^4 + y^3}$

15. $\dfrac{x^{12} - y^8}{x^6 + y^4}$

16. In Ex. 11 remove the parenthesis in the dividend and divisor, and divide by long division. The work required is about how many times that required in the abbreviated process?

Write a binomial divisor and the corresponding quotient for each of the following:

17. $\dfrac{b^3 - 8y^3}{\qquad} =$

18. $\dfrac{b^2 - 4y^2}{\qquad} =$

19. $\dfrac{b^3 + 8y^3}{\qquad} =$

20. $\dfrac{8x^3 - 27}{\qquad} =$

21. $\dfrac{8x^3 + 27}{\qquad} =$

22. $\dfrac{4x^2 - 9}{\qquad} =$

23. $\dfrac{a^5 - 1}{\qquad} =$

24. $\dfrac{a^2 - 4(x + y)^2}{\qquad} =$

25. $\dfrac{a^3 + 8(x + y)^3}{\qquad} =$

26. $\dfrac{a^3 - 8(x + y)^3}{\qquad} =$

27. $\dfrac{8x^6 + y^9}{\qquad} =$

28. $\dfrac{4x^4 - y^6}{\qquad} =$

29. $\dfrac{32a^5 - y^5}{\qquad} =$

30. $\dfrac{8x^6 - y^9}{\qquad} =$

Divide each of the following by $x - a$ in a short way:

31. $x^3 - a^3 + x^2 - a^2$

32. $x^3 - a^3 + 5(x^2 - a^2)$

33. $x^3 - a^3 + 5(x - a)$

34. $3(x^3 - a^3) + 4(x - a)$

35. $(x - a)^3 + 5(x^2 - a^2)$

36. $7(x - a)^3 + 5(x - a)$

Find the value of each of the following in the shortest way:

37. $(a + b)(a + b)(a - b)(a - b)$

38. $(a + 2b)(a + 2b)(a - 2b)(a - 2b)$

39. $(3x - 2y)(3x - 2y)(3x + 2y)(3x + 2y)$

Simplify:

40. $5x - 3(x - 2)^2 - 3(3 - 2x)(1 + x)$

41. $7 - 5(x - 2)^2 - 3(3 - 2x)(-x)$

Solve:

42. $(x - 8)(x + 12) - (x + 1)(x - 6) = 0$

43. $(2x - 1)(x + 3) - (x - 3)(2x - 3) = 72$

44. $\dfrac{x^2 - 4}{x - 2} + 3x = 8$

45. $\dfrac{x^3 - 8}{x - 2} - x^2 + 4 = 0$

46. Four times a certain number diminished by 12.07, equals twice the number increased by 1.13. Find the number.

47. Separate 1000 into three parts such that the second part is three times as large as the first part, and the third part exceeds the first part by 100.

48. The Suez Canal is 100 miles long. This is 2 miles more than 8 times the length of the Simplon Tunnel. Find the length of the tunnel.

49. The temperature of the electric arc is 5400° F. This is 464° more than 8 times the temperature at which lead fuses. Find the temperature at which lead fuses.

50. The velocity of sound in the air is 1090 ft. per second. This rate is 10 ft. more than 9 times the rate at which sensation travels along a nerve. Find the rate at which sensation travels. How does this compare with the velocity of an express train going 60 miles per hour?

51. Who first used the sign + to denote addition, and when? (See p. 457.)

52. Give some other symbols used to represent addition before the sign + was invented. Discuss as far as you can the relative advantages of these signs.

53. Answer the questions in Ex. 52 for the subtraction sign.

54. Answer the questions in Ex. 52 for the sign ×.

55. Answer the questions in Ex. 52 for the sign ÷.

56. Answer the questions in Ex. 52 for the sign =.

CHAPTER VIII

FACTORING

85. The Factors of an expression (see Art. 11) are the quantities which, multiplied together, produce the given expression.

Factoring is the process of separating an algebraic expression into its factors.

86. Utility of Factoring. If it is known that

$$x^2 - 8x + 15 = (x - 3)(x - 5)$$

and

$$2x^2 - 13x + 21 = (2x - 7)(x - 3)$$

Then

$$\frac{x^2 - 8x + 15}{2x^2 - 13x + 21} = \frac{(x - 3)(x - 5)}{(x - 3)(2x - 7)} = \frac{x - 5}{2x - 7}$$

The above reduction of a fraction to a simpler form illustrates the usefulness of a knowledge of factoring in enabling us to simplify work and save labor.

Why do we now proceed to make definitions and rules and to divide the topic, Factoring, into cases?

87. A Prime Quantity in algebra is one which cannot be divided by any quantity except itself and unity; as a, b, $a^2 + b^2$, 17.

In all work in factoring, prime factors are sought, unless the contrary is stated.

88. Perfect Square and Perfect Cube. When an expression is separable into two equal factors, the expression is called a *perfect square,* and each of the factors is the *square root* of the expression.

Thus, $9a^2x^4 = 3ax^2 \cdot 3ax^2$.

$\therefore 3ax^2$ is the square root of $9a^2x^4$.

Also, $x^2 - 4x + 4 = (x - 2)(x - 2)$, and is therefore a perfect square, with $x - 2$ for its square root.

When an expression is separable into three equal factors, the expression is called a *perfect cube,* and each of the factors is its *cube root.*

Thus, $27a^3x^6y^9 = 3ax^2y^3 \cdot 3ax^2y^3 \cdot 3ax^2y^3$.

$\therefore 3ax^2y^3$ is the cube root of $27a^3x^6y^9$.

89. The Factors of a Monomial are recognized by direct inspection.

Thus, the factors of $7a^2x^3$ are 7, a, a, x, x, x.

90. Factors of Polynomials. Multiply $4x^2 + 2xy + y^2$ by $4x^2 - 2xy + y^2$. What terms are canceled in adding the partial products? Because these terms have been thus canceled and have disappeared, it is difficult to take the final product $16x^4 + 4x^2y^2 + y^4$ and from it discover the original factors which were multiplied together to produce it.

Hence, in factoring polynomials various methods must be devised to meet different cases, and the cases must be carefully discriminated.

CASE I

91. A Polynomial having a Common Factor in all its Terms.

Ex. Factor $3x^2 + 6x$.

$$3x^2 + 6x = 3x(x + 2) \quad \textit{Factors}$$

At first, in working examples of this kind, it is well to put the work in the following form:

$$3x\underline{)3x^2 + 6x} = 3x(x + 2) \quad \textit{Factors}$$
$$x + 2$$

Check by Substitution If we let $x = 2$

$$3x(x+2) = 6(2+2) = 24$$

Also $$3x^2 + 6x = 12 + 12 = 24$$

Check by Multiplication

$$\begin{array}{l} x + 2 \\ 3x \\ \hline 3x^2 + 6x \end{array}$$

Hence, in general,

Divide all the terms of the polynomial by their largest common factor;

The factors will be the divisor and quotient.

EXERCISE 36

Factor the following and check the work for each example either by substitution or by multiplication, or by both as the teacher may direct:

1. $2x^3 + 5x^2$

2. $x^3 - 2x$

3. $x^2 + x$

4. $3a^2 - a$

5. $7a + 14a^3$

6. $3a^3x^2 - 15a^2x^3$

7. $18x^5 - 27x^4y$

8. $x^2 - x^3 - x^4$

9. $a^2x - 2a^2x^2$

10. $\frac{1}{2}x^3 + \frac{1}{4}x^2$

11. $\frac{3}{8}a^2b - \frac{5}{2}ab^2$

12. $\frac{5}{6}ax^3 - 2x^4$

13. $.2x^3 + .4ax^2$

14. $.02ax^2 - .4a^2x^2$

15. $1.2mn - .6m^2$

16. $3a^2 - 6ax + 9x^2$

17. $2x + 4x^2 - 6x^3$

18. $10a^3b^2 - 35a^2b^3$

19. $a^4b^3c - a^3b^3c^3 + 2a^2b^4c^2$

20. $2x^ny^4 - 8x^{2n}y^3 + 6x^{3n}y^2$

21. $a^mb^3c^{2n} + 11a^mb^3c^{2n+1}$

22. $7(a+b)x + 5(a+b)y$

23. $7(a+b)x^2y + 5(a+b)^2xy^2$

24. $21(x-y)^2 - 14(x-y)^3$

25. $9(2x-a)^3 - 12(2x-a)^5$

In the shortest way find the value of

26. $847 \times 915 - 847 \times 913$

27. $312.75 \times 87 - 312.75 \times 84$

28. $8 \times 11 \times 23^2 + 7 \times 11 \times 23^2 - 5 \times 11 \times 23^2$

29. $\pi R^2 + \pi r^2$ when $\pi = \frac{22}{7}$, $R = 8$, and $r = 6$.

30. $\pi R^2 - \pi r^2$ when $\pi = \frac{22}{7}$, $R = 410$, and $r = 60$.

Find the value of x in the following equations:

31. $ax + bx = 10$. (What does the value of x become when $a = 5$ and $b = 15$?)

32. $ax = 10 - bx$

33. $ax + bx + cx = 12$

34. $2ax - bx + 3cx = 15$

Factor the numerator and denominator of each of the following fractions and then simplify the fraction by canceling factors:

35. $\dfrac{3a^2b - 6ab^2}{3a^2b + 6ab^2}$

36. $\dfrac{x^3 + 2x^2}{3x^4 - 6x^2}$

37. $\dfrac{a^2x^3}{4x^4 - 6x^3}$

38. $\dfrac{3pq - 6p^2q^2}{12p^3q^3 - 6pq}$

39. From an examination of Exs. 26–38, state the uses or advantages of being able to factor by the method of Case I.

40. How many of the examples in this Exercise can you work at sight?

Case II

92. A Trinomial that is a Perfect Square. By Arts. 75 and 76 a trinomial is a perfect square when its first and last terms are perfect squares and positive, and the middle term is twice the product of the square roots of the end terms. The sign of the middle term determines whether the square root of the trinomial is a sum or a difference.

Ex. 1. Factor $16x^2 - 24xy + 9y^2$.

$$16x^2 - 24xy + 9y^2 = (4x - 3y)(4x - 3y) \textit{ Factors}$$

Ex. 2. Factor $(a + b)^2 + 4(a + b)x + x^2$.

$$\begin{aligned}(a + b)^2 + 4(a + b)x + 4x^2 &= [(a + b) + 2x]^2 \\ &= (a + b + 2x)^2 \textit{ Ans.}\end{aligned}$$

Hence, in general, to factor a trinomial that is a perfect square,

Take the square roots of the first and last terms, and connect these by the sign of the middle term;

Take the result as a factor twice.

EXERCISE 37

Factor and check:

1. $4x^2 + 4xy + y^2$
2. $16a^2 - 24ay + 9y^2$
3. $25x^2 - 10x + 1$
4. $x^2 - 20xy + 100y^2$
5. $49c+28bc^2 + 4b^2c^3$
6. $a^3b^2 + 4a^3b + 4a^3$
7. $xy^2 + 2xy + x$
8. $2m^2n - 4mn + 2n$
9. $a^5 + 2a^4 + a^3$
10. $4x^3 + 44x^2y^2 + 121xy^4$
11. $81a^3b + 126a^2b^2 + 49ab^3$
12. $8a^2y - 40axy + 50x^2y$
13. $2x^4 - 8x^3 + 8x^2$
14. $30x^2y + 3x^4 + 75y^2$
15. $a^3x + ax^3 - 2a^2x^2$
16. $x^{2n} + 2x^ny + y^2$
17. $(a - b)^2 - 2c(a - b) + c^2$
18. $9(x + y)^2 + 12z(x + y) + 4z^2$
19. $16(2a - 3)^2 - 16ab + 24b + b^2$
20. $25(x - y)^2 - 120xy(x - y) + 144x^2y^2$
21. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

22. $\frac{4}{9}x^2 + 4xy + 9y^2$

23. $\frac{1}{4}x^2 + \frac{1}{3}xy + \frac{1}{9}y^2$

24. $.04a^2 - .12ab + .09b^2$

25. $25a^2 - 30ax + 9x^2$

Solve the following equations for x:

26. $ax - 3bx = a^2 - 6ab + 9b^2$

27. $ax + 3bx = a^2 + 6ab + 9b^2$

28. $x - 2ax = 1 - 4ax + 4a^2$

29. $2ax + 3bx = 4a^2 + 12ab + 9b^2$

Factor the following examples in Cases I and II, and check each result:

30. $x^3 - 4x^2 + 4x$

31. $x^3 - 3x^2 + 4x$

32. $m^5 - 2m^4n + m^3n^2$

33. $m^5 - m^4n + m^3n^2$

34. $a^3x^2 - 8a^3x + 16a^3$

35. $a^2x^2 - 6a^2x + 4a^2$

36. $10a^2 - 20a + 10$

37. $20a^2 - 10a + 10$

38. $16a^2p^2 - 24a^2p + 9a^2$

39. $4x^4 + 4x^3y + x^2y^2$

40. $8x^3 + 16x^2y + xy^4$

41. $16m^2n^3 - 9mn^3 + n^3$

42. How many of the examples in this Exercise can you work at sight?

CASE III

93. The Difference of Two Perfect Squares.

From Art. 77, p. 112, $(a + b)(a - b) = a^2 - b^2$

Hence, $a^2 - b^2 = (a + b)(a - b)$

But any algebraic quantities may be used instead of a and b. Hence, in general, to factor the difference of two squares,

Take the square root of each square;

The factors will be the sum of these roots and their difference.

Ex. 1. Factor $x^2 - 16y^2$.

$$x^2 - 16y^2 = (x + 4y)(x - 4y) \quad \textit{Factors}$$

Ex. 2. Factor $x^4 - y^4$.

$$\begin{aligned} x^4 - y^4 &= (x^2 + y^2)(x^2 - y^2) \\ &= (x^2 + y^2)(x + y)(x - y) \quad \textit{Factors} \end{aligned}$$

Ex. 3. Factor $(3x + 4y)^2 - (2x + 3y)^2$.

$$\begin{aligned} (3x + 4y)^2 - (2x + 3y)^2 &= [(3x + 4y) + (2x + 3y)]\,[(3x + 4y) - (2x + 3y)] \\ &= (3x + 4y + 2x + 3y)(3x + 4y - 2x - 3y) \\ &= (5x + 7y)(x + y) \quad \textit{Factors} \end{aligned}$$

Let the pupil check the above examples.

EXERCISE 38

Factor and check:

1. $x^2 - 9$

2. $25 - 16a^2$

3. $4a^2 - 49b^2$

4. $x^4 - 4y^2$

5. $100 - 81m^2$

6. $9a^4 - x^6$

7. $1 - 64m^2$

8. $3x^2 - 12y^2$

9. $x^3 - 9a^2x$

10. $x^4 - 9a^2x^2$

11. $m - 64a^2m$

12. $242 - 2x^2$

13. $x^5 - x^3$

14. $3x^3 - 75xy^6$

15. $a^4 - x^4$

16. $a^4 - 81b^4$

17. $x^8 - y^8$

18. $x^5 - x$

19. $ax^6 - ax^2$

20. $a^8x - x$

21. $225x^{2n} - y^2$

22. $2\frac{1}{4}x^2 - \frac{4}{9}y^2$

23. $\frac{4}{25}a^2 - 9b^2$

24. $.09x^2 - .16y^2$

25. $.01a^2 - .04b^2$

26. $.25y^2 - \frac{1}{25}b^2$

27. $.81x^2 - .0025b^2$

28. $x^6 - y^6$

29. $x^{4n} - y^{2n}z^6$

30. $(x + y)^2 - 1$

31. $x^2 - (y + 1)^2$

32. $(x - y)^2 - 9$

33. $4(x - y)^2 - 25$

34. $1 - 36(x + 2y)^2$

35. $(x + 2y)^2 - (3x + 1)^2$

36. $25(2a - b)^2 - (a - 3b)^2$

37. $x^{12}y^9 - yz^{16}$

38. $81x^{12} - 16y^4$

39. $x^5 - 144xy^2z^6$

40. $(a - b)^2 - 4(c + 1)^2$

41. $1 - 100(x^2 - x - 1)^2$

Solve the following equations for x:

42. $ax + 2bx = a^2 - 4b^2$
44. $3x - ax = 9 - a^2$
43. $ax - 2bx = a^2 - 4b^2$
45. $x - bx = 1 - b^2$

Factor the following miscellaneous examples:

46. $a^2 - 4a$
47. $a^2 - 4$
48. $a^2 - 4a + 4$
49. $a^3 - 4a$
50. $a^3 - 4a^2 + 4a$
51. $a^4 - 4a^3 + 4a^2$
52. $x^3 - 6x$
53. $x^3 - 9x$
54. $a^3 + 9a^2x + 6ax^2$
55. $a^3 + 6a^2x + 9ax^2$
56. $a^5 - ax^4$
57. $a^5 + ax^4$
58. $(a + x)^2 - 9$
59. $(a + b)^2 - (x - y - a)^2$

60. Make up and work an example in factoring in each of the cases treated thus far.

61. How many of the examples in this Exercise can you work at sight?

62. How many of the examples in Exercise 2 (p. 13) can you now work at sight?

CASE IV

94. A Trinominal of the Form $x^2 + bx + c$.

It was found in Art. 79 (p. 117) that on multiplying two binomials like $x + 3$ and $x - 5$, the product, $x^2 - 2x - 15$, was formed by taking the algebraic sum of $+ 3$ and $- 5$ for the coefficient of x (viz. $- 2$), and taking their product ($- 15$) for the last term of the result. Hence, in undoing this work to find the factors of $x^2 - 2x - 15$, the essential part of the process is to find two numbers which, added together, will give $- 2$ and, multiplied together, will give $- 15$.

Ex. 1. Factor $x^2 + 11x + 30$.

The pairs of numbers whose product is 30 are 30 and 1, 15 and 2, 10 and 3, 6 and 5. Of these, that pair whose sum is 11 is 6 and 5.

Hence, $x^2 + 11x + 30 = (x + 6)(x + 5)$ *Factors*

Ex. 2. Factor $x^2 - x - 30$.

It is necessary to find two numbers whose product is -30, and whose sum is -1.

Since the sign of the last term is minus, the two numbers must be one positive and the other negative; and since their sum is -1 the greater number must be negative.

$$x^2 - x - 30 = (x - 6)(x + 5) \textit{ Factors}$$

Ex. 3. Factor $x^2 + 3xy - 10y^2$.

Since $5y$ and $-2y$, added give $3y$, and multiplied give $-10y^2$,

$$x^2 + 3xy - 10y^2 = (x + 5y)(x - 2y) \textit{ Factors}$$

Hence, in general, to factor a trinomial of the form

$$x^2 + bx + c,$$

Find two numbers which, multiplied together, produce the third term of the trinomial and, added together, give the coefficient of the second term;

x (or whatever takes the place of x) plus the one number, and x plus the other number, are the factors required.

EXERCISE 39

Factor and check:

1. $x^2 + 5x + 6$
2. $x^2 - x - 6$
3. $x^2 + x - 6$
4. $x^2 + 7x - 44$
5. $x^2 - 11x + 30$
6. $x^2 + x - 30$
7. $x^2 + 6xy - 16y^2$
8. $x^2 - 6xy - 16y^2$
9. $x^2 + 8x + 16$
10. $x^2 + 5x - 36$

11. $x^2 - 5x - 36$

12. $x^4 - 5x^2 - 36$

13. $x^2 + 3x - 28$

14. $x^2 - 2x - 48$

15. $x^2 - 8x - 48$

16. $x^2 + 13x - 48$

17. $x^2 - 22x - 48$

18. $x^2 - 4x - 96$

19. $x^2y^2 - 23xy + 132$

20. $x^2 - 5ax - 24a^2$

21. $x^4 - 9x^2 + 8$

22. $2a - 14ax - 60ax^2$

23. $2x^3 - 22x^2 - 120x$

24. $x^5 - 25x^3 + 144x$

25. $x^{2n} - x^n - 56$

26. $a^2b^2 - 11abc^2 - 26c^4$

27. $x^2 + (a + b)x + ab$

28. $x^2 + (2a - 3b)x - 6ab$

29. $x^2 + (a + 2b + c)x + (a + b)(b + c)$

30. $x^2 + (a + b)x + (a - c)(b + c)$

31. $(x - y)^2 - 3(x - y) - 18$

Factor and check each of the following miscellaneous examples:

32. $x^2 - 4x + 4$

33. $x^2 - 4$

34. $x^2 - 4x + 3$

35. $x^3 - x^2 - 6x$

36. $x^3 - 4x$

37. $x^3 + 6x^2 + 9x$

38. $a^4 - 4y^2$

39. $a^4 - 4a^2y + a^2y^4$

40. $a^8 - 1$

41. $x^2 + 5ax + 6a^2$

42. $x - x^5$

43. $a^4 - 7a^2 + 12$

44. Make up and work an example in factoring to illustrate each case treated thus far.

45. How many of the examples in this Exercise can you work at sight?

CASE V

95. A Trinomial of the Form $ax^2 + bx + c$.

From Art. 80 (p. 119) it is evident that the essential part of the process of factoring a trinomial of the form $ax^2 + bx + c$ lies in determining two factors of the first term and two factors of the last term, such that the algebraic sum of the cross products of these factors equals the middle term of the trinomial.

Ex. Factor $10x^2 + 13x - 3$.

The possible factors of the first term are $10x$ and x, $5x$ and $2x$. The possible factors of the third term are -3 and 1, 3 and -1. In order to determine which of these pairs will give $+13x$ as the sum of their cross products, it is convenient to arrange the pairs thus:

$$\begin{array}{ccc} 10x, & & -3 \\ & \times & \\ x, & & 1 \end{array} \qquad\qquad \begin{array}{ccc} 5x, & & -1 \\ & \times & \\ 2x, & & 3 \end{array}$$

Variations may be made mentally by transferring the minus sign from 3 to 1; and also by interchanging the 3 and the 1.

It is found that the sum of the cross products of

$$\begin{array}{cc} 5x, & -1 \\ 2x, & 3 \end{array} \quad \text{is } +13x$$

Hence, $\quad 10x^2 + 13x - 3 = (5x - 1)(2x + 3)$ *Factors*

Let the pupil check the work.

Hence, in general, to factor a trinomial of the form

$$ax^2 + bx + c,$$

Separate the first term into two such factors, and the third term into two such factors, that the sum of their cross products equals the middle term of the trinomial;

As arranged for cross multiplication, the upper pair taken together and the lower pair taken together form the two factors.

EXERCISE 40

Factor and check:

1. $2x^2 + 3x + 1$
2. $3x^2 - 14x + 8$
3. $2x^2 + 5x + 2$
4. $3x^2 + 10x + 3$
5. $6x^2 - 7x - 5$
6. $2x^2 + 5x - 3$
7. $6x^3 + 20x^2 - 16x$
8. $3x^4 - 4x^3 - 4x^2$
9. $8a^2 + 2a - 15$
10. $2x^2 + x - 10$
11. $12x^2 - 5x - 2$
12. $4x^2 + 11x - 3$
13. $5x^2 + 24x - 5$
14. $9x^3 - 15x^2 - 6x$
15. $6x^2y - 2xy - 4y$
16. $16x^2 - 6xy - 27y^2$
17. $12x^2 + xy - 63y^2$
18. $32a^2 + 4ab - 45b^2$
19. $4x^4 - 13x^2 + 9$
20. $9x^4 - 148x^2 + 64$
21. $12x^2 - 7xz - 12z^2$
22. $24x^3 + 104x^2y^2 - 18xy^4$
23. $25a^4 + 9a^2b^2 - 16b^4$
24. $16x^4 - 10x^2y^2 - 9y^4$
25. $3x^{2n} - 8x^ny - 3y^2$
26. $25a^4 - 41a^2b^2 + 16b^4$
27. $20 - 9x - 20x^2$
28. $5 + 32xy - 21x^2y^2$
29. $(a + b)^2 + 5(a + b) - 24$
30. $3(x - y)^2 + 7(x - y)z - 6z^2$
31. $3(x^2 + 2x)^2 - 5(x^2 + 2x) - 12$
32. $4x(x^2 + 3x)^2 - 8x(x^2 + 3x) - 32x$
33. $2(x + 1)^2 - 5(x^2 - 1) - 3(x - 1)^2$

Factor and check each of the following miscellaneous examples:

34. $4x^2 - 1$
35. $4x^2 + 4x + 1$
36. $3x^2 + 4x + 1$
37. $x^2 + 4x + 3$
38. $x^4 - 1$
39. $x^3 - x^2 - 6x$
40. $5a^2 + 9a - 2$
41. $x^2 - 9x + 18$

42. $x^3 - 6x^2 + 9x$

43. $a^2 - 4(x + y)^2$

44. $3x^2 + 7x - 6$

45. $(a + b)^2 + 2(a + b)x + x^2$

46. $x^8 - a^8$

47. $x^7 - x$

48. $x^2 - ax - 2a^2$

49. $2x^3 - 5x^2 - 3x$

50. Make up and work an example in factoring in each case treated thus far.

51. How many of the examples in this Exercise can you work at sight?

52. How many examples in Exercise 25 (p. 101) can you work at sight?

CASE VI

96. Sum or Difference of Two Cubes.

From Art. 83 (p. 125), $\dfrac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$

Hence, $a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad . \; . \; . \; (1)$

In like manner, $a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad . \; . \; . \; (2)$

But any algebraic expressions may be used instead of a and b in formulas (1) and (2).

Ex. 1. Factor $27x^3 - 8y^3$.

$$27x^3 - 8y^3 = (3x)^3 - (2y)^3$$

Use $3x$ for a and $2y$ for b in (2) above.

$$27x^3 - 8y^3 = (3x - 2y)(9x^2 + 6xy + 4y^2) \textit{ Factors}$$

In working examples of this type, it is often convenient to call $3x - 2y$ the "divisor factor" and $9x^2 + 6xy + 4y^2$ the "quotient factor." Why are these names appropriate in this case?

Ex. 2. Factor $a^6 + 8b^9$.

$$\begin{aligned} a^6 + 8b^9 &= (a^2)^3 + (2b^3)^3 \\ &= (a^2 + 2b^3)(a^4 - 2a^2b^3 + 4b^6) \textit{ Factors} \end{aligned}$$

Ex. 3. Factor $(a+b)^3 - x^3$.

$$(a+b)^3 - x^3 = [(a+b) - x]\,[(a+b)^2 + (a+b)x + x^2]$$

Let the pupil check the above examples.

Hence, in general, to factor the sum or difference of two cubes,

Obtain the values of **a** *and* **b** *in the given example, and substitute these values in formula* (1) *or* (2).

97. Sum or Difference of Two Like Odd Powers.

Since the *difference* of two like *odd* powers is always divisible by the difference of their roots (see Art. 84, p. 128), the factors of $a^n - b^n$, when n is odd, are the divisor, $a - b$, and the quotient.

Ex. 1. Factor $a^5 - b^5$.

$$a^5 - b^5 = (a-b)\,(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$$

Since the *sum* of two like *odd* powers is divisible by the sum of the roots (see Art. 84, p. 128), the factors of $a^n + b^n$, when n is odd, are the divisor, $a + b$, and the quotient.

Ex. 2. Factor $x^5 + 32y^5$.

$$\begin{aligned} x^5 + 32y^5 &= x^5 + (2y)^5 \\ &= (x+2y)\,[x^4 - x^3(2y) + x^2(2y)^2 - x(2y)^3 + (2y)^4] \\ &= (x+2y)\,(x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4). \ \textit{Factors} \end{aligned}$$

98. Sum or Difference of Two Like Even Powers.

The *difference* of two like *even* powers is factored to best advantage by Case III (p. 139).

Ex. 1. $x^8 - y^8$.

$$= (x^4 + y^4)\,(x^2 + y^2)\,(x + y)\,(x - y) \quad \textit{Factors}$$

The *sum* of two like *even* powers cannot in general be factored by elementary methods unless the expression may be regarded as the sum or difference of two cubes (Art. 96), or other like odd powers.

Ex. 2. $a^6 + b^6 = (a^2)^3 + (b^2)^3$

$$= (a^2 + b^2)\,(a^4 - a^2b^2 + b^4)\ \textit{Factors}$$

But $a^2 + b^2$, $a^4 + b^4$, and $a^8 + b^8$ cannot be factored by any elementary method, and are therefore prime expressions.

Let the pupil check the examples of Art. 97 and 98.

EXERCISE 41

Factor and check:

1. $m^3 - n^3$
2. $c^3 + 8d^3$
3. $27 - x^3$
4. $a^3 + 8b^3c^3$
5. $x^3 - 125$
6. $64y^3 - 27$
7. $a^3b^3 + 1$
8. $1 - 1000x^3$
9. $27x^4 + a^3x$
10. $512x^3 - y^6$
11. $a + 343a^4$
12. $a^6 - x^6$
13. $x^{12} - y^6$
14. $a^6 - 64n^{12}$
15. $250x - 2x^7$
16. $8x^6 + y^3$
17. $(a + b)^3 + 1$
18. $125 + (2b - a)^3$
19. $8 - (c + d)^3$
20. $(x - y)^3 - 27x^3$
21. $16x^4y^6 - 54xz^3$
22. $x^5 + y^5$
23. $x^7 - y^7$
24. $a^6 + m^6$
25. $x^{12} + y^{12}$
26. $a^7 - 128b^7$
27. $a^{11} + x^{11}$
28. $a^9 + b^9$
29. $32x^5 - 1$
30. $a^{11} - b^{11}$
31. $243 - x^5$
32. $64 - (a - b)^3$
33. $8(x - 2y)^3 + 1$
34. $a^{10} - b^{10}$
35. $a^{10} + b^{10}$
36. $32x^5 - a^{10}$
37. $a^6 + y^9$
38. $8x^{12} + y^9$
39. $512x^3 - (a + b)^6$

40. Make up a binomial expression whose terms contain unlike exponents and which can be factored as the sum of two cubes. Also one that can be factored as the sum of two 5th powers.

41. Make up a binomial the exponents in whose terms are even numbers, and which can be factored as the sum of

two cubes. Also one that can be factored as the sum of two 5th powers.

42. State which of the following expressions can be factored:

$$\begin{array}{llll} x^3 + y^8 & x^3 + y^5 & x^3 - y^{10} & x^6 + y^{10} \\ x^3 + y^9 & x^3 - y^{12} & x^6 + y^8 & x^6 + y^{12} \\ x^3 + y^6 & x^6 - y^9 & x^6 + y^9 & x^5 - y^{10} \end{array}$$

Factor and check each of the following miscellaneous examples:

43. $x^2 - 4a^2$

44. $x^3 - 8a^3$

45. $x^2 - 4ax + 4a^2$

46. $x^2 - 4ax + 3a^2$

47. $x^4 - a^4$

48. $x^5 + a^5$

49. $x^2 - 4(a + b)^2$

50. $x^3 - 8(a + b)^3$

51. $x^6 + a^9$

52. $x^6 - a^6$

53. $x^{12} + y^9$

54. $x^{12} - y^{12}$

55. $6a^2 - 13a + 6$

56. $16x^2 - 8xy + y^2$

57. $x^4 + 27a^3x$

58. $x^{12} + y^3$

59. Make up and work an example in factoring to illustrate each case treated thus far.

60. How many of the examples in this Exercise can you work at sight?

CASE VII

99. A Polynomial whose Terms may be Grouped so as to be Divisible by a Binomial Divisor.

Ex. 1.
$$\begin{aligned} ax - ay - bx + by &= (ax - ay) - (bx - by) \\ &= a(x - y) - b(x - y) \\ &= (a - b)(x - y) \ \textit{Factors} \end{aligned}$$

The last step in the preceding process is sometimes clearer to the pupil when written in the following form:

$$(x - y)\,\overline{)a(x - y) - b(x - y)} \atop \quad\; a \qquad\quad -b \quad = (x - y)(a - b) \ \textit{Factors}$$

Ex. 2. $1 + 15a^4 - 5a - 3a^3 = 1 - 3a^3 - 5a + 15a^4$
$= (1 - 3a^3) - 5a(1 - 3a^3)$
$= (1 - 3a^3)(1 - 5a)$ *Factors*

Ex. 3. $a^3 + 3a^2 - 4 = a^3 + 2a^2 + a^2 - 4$
$= a^2(a + 2) + (a + 2)(a - 2)$
$= (a + 2)(a^2 + a - 2)$
$= (a + 2)(a + 2)(a - 1)$ *Factors*

Let the pupil check the above examples.

EXERCISE 42

Factor and check:

1. $ax + ay + bx + by$
2. $x^2 - ax + cx - ac$
3. $5xy - 10y - 3x + 6$
4. $3am - 4mn - 6ay + 8ny$
5. $a^2x + 3ax + acx + 3cx$
6. $3a^2 + 3ab - 5an - 5bn$
7. $x^4 + x^3 + 2x^2 + 2x$
8. $2x^4 - 2x^3 - 2a^2x^2 + 2a^2x$
9. $y^3 + y^2 + y + 1$
10. $ax^2 - 2a^2x - x + 2a$
11. $x^2 + 3y - 3x - xy$
12. $z^3 - z^2 - z + 1$
13. $ab - by - a + y$
14. $x^5 - x^4 - 4x + 4$
15. $a^2x^2 - b^2x^2 - a^2y^2 + b^2y^2$
16. $x(x + 4)^2 + 4(x + 4)$
17. $a^2(a + 3) - 3(a + 3)$
18. $2(x^2 - y^2) - (x - y)$
19. $4x(x - 1)^2 + x - 1$
20. $x^3 - 1 + 2(x^2 - 1)$
21. $x^2 - y^2 + x^3 - y^3$
22. $x - y + x^3 - y^3$
23. $x^3 - y^3 - x^2 + y^2$
24. $x^3 - y^3 - (x - y)^2$
25. $4a^2 - a^2x^2 + x^2 - 4$
26. $x^3 - y^3 + x^2 - y^2 + x - y$
27. $4ax^3 + 8ax - 8a - 4ax^2$
28. $a(3a - x)^2 - 6ax^2 + 2x^3$
29. $x^3 - 8 - 7(x - 2)$
30. $4(x^3 + 27) - 31x - 93$
31. $(2x + 1)^3 - (2x + 1)(3x + 4)$
32. $(2x - 3)^3 + 2x^2 - 9x + 9$

33. $x^3 - 7x - 6$

34. $x^3 - 3x^2 - 10x + 24$

35. $x^3 - 8x^2 + 17x - 10$

Factor and check each of the following miscellaneous examples:

36. $a^3 - 8x^3$

37. $a^2 - 16x^2$

38. $a^2 - 6ax + 9x^2$

39. $ax - bx + ay - by$

40. $a^2 - a - 2$

41. $x^3 - a^3 + 2(x - a)$

42. $3a^2 - 4a - 4$

43. $x^6 + y^9$

44. $x^3 - x^2 + x - 1$

45. $x^3 - 9x^2 + 18x$

46. $x^6 + (a + b)^3$

47. $a^8 - y^8$

48. $(a + b)^2 - 2(a + b)p + p^2$

49. $(a + b)^2 - (a + b) - 2$

50. $x^2 + x^3 - a^2 - a^3$

51. $x + a^3 - a - x^3$

52. Make up and work an example to illustrate each case in factoring treated thus far.

53. How many of the examples in this Exercise can you work at sight?

SPECIAL CASES UNDER CASE III

100. By the **Grouping of Terms** we may often reduce an expression to the difference of two perfect squares.

Ex. 1. Factor $x^2 - 4xy + 4y^2 - 9z^2$.

$$\begin{aligned} x^2 - 4xy + 4y^2 - 9z^2 &= (x^2 - 4xy + 4y^2) - 9z^2 \\ &= (x - 2y)^2 - 9z^2 \\ &= [(x - 2y) + 3z]\,[(x - 2y) - 3z] \\ &= (x - 2y + 3z)\,(x - 2y - 3z) \quad \textit{Factors} \end{aligned}$$

Ex. 2. Factor $a^2 - x^2 - y^2 + b^2 + 2ab + 2xy$.

$$\begin{aligned} a^2 - x^2 - y^2 + b^2 + 2ab + 2xy &= (a^2 + 2ab + b^2) - (x^2 - 2xy + y^2) \\ &= (a + b)^2 - (x - y)^2 \\ &= (a + b + x - y)\,(a + b - x + y) \\ &\qquad \textit{Factors} \end{aligned}$$

Let the pupil check the above examples.

EXERCISE 43

Factor and check:

1. $a^2 + 2ab + b^2 - x^2$
2. $a^2 - 2ab + b^2 - 4x^2$
3. $a^2 - x^2 - 2xy - y^2$
4. $9a^2 - x^2 - 4xy - 4y^2$
5. $16a^2 - x^2 + 2xy - y^2$
6. $m^2 - x^2 - y^2 - 2xy$
7. $a^2 + b^2 + 2ab - 4x^2$
8. $a^2 + b^2 - 4x^2 + 2ab$
9. $a^2 - 4x^2 + b^2 + 2ab$
10. $x^2 - 2ab - a^2 - b^2$
11. $2ab + x^2 - a^2 - b^2$
12. $x^2 + a^2 - y^2 - 2ax$
13. $a^2 + y^2 - x^2 + 2ay$
14. $a^4 - x^4 - 2x^2y - y^2$
15. $x^2 - y^2 - 1 - 2y$
16. $1 + 2xy - x^2 - y^2$
17. $c^2 - a^2 - b^2 + 2ab$
18. $a^2 + b^2 - c^2 - 2ab$
19. $2ab + a^2b^2 + 1 - x^2$
20. $2z^2 - 4z - 2z^4 + 2$
21. $20yz + x^2 - 4y^2 - 25z^2$
22. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$
23. $x^2 + 4y^2 - 9z^2 - 1 - 4xy - 6z$
24. $9a^2 - 25x^2 + 4b^2 - 1 - 10x - 12ab$
25. $a^2 - 9b^2x^2 - 1 + 6bx - 10ab + 25b^2$

Factor and check each of the following miscellaneous examples:

26. $ax - bx + ay - by$
27. $a^2 - x^2 - 2xy - y^2$
28. $a^2 + ax - ab - bx$
29. $a^2 - 2ab + b^2 - x^2$
30. $2a + 2b - 3a - 3b$
31. $4a^2 + 4a + 1 - b^2$
32. $9x^2 - 4a^2 - 4ab - b^2$
33. $9x^2 - 9y^2 + x - y$
34. $x^3 - b^3 + x^2 - b^2$
35. $a^2 + b^2 - y^2 + 2ab$
36. $a^3 - 27y^3$
37. $a^2 - 6ay + 9y^2$
38. $a^4x - 16xy^4$
39. $3a^2 - 4a + 1$
40. $x^6 - 8(a + b)^3$
41. $a^6 + y^9$

42. Make up and work an example in each case in factoring treated thus far.

43. How many of the examples in this Exercise can you work at sight?

101. The Addition and Subtraction of a Square will sometimes transform a given expression into a difference of two perfect squares.

Ex. 1. Factor $a^4 + a^2b^2 + b^4$.

Add and subtract a^2b^2.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \quad \textit{Factors} \end{aligned}$$

Ex. 2. Factor $x^4 - 7x^2y^2 + y^4$.

Add and subtract $9x^2y^2$.

$$\begin{aligned} x^4 - 7x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - 9x^2y^2 \\ &= (x^2 + y^2)^2 - 9x^2y^2 \\ &= (x^2 + y^2 + 3xy)(x^2 + y^2 - 3xy) \quad \textit{Factors} \end{aligned}$$

Let the pupil check the above examples.

EXERCISE 44

Factor and check:

1. $c^4 + c^2x^2 + x^4$

2. $x^4 + x^2 + 1$

3. $4x^4 - 13x^2 + 1$

4. $4a^4 - 21a^2b^2 + 9b^4$

5. $9x^4 + 3x^2y^2 + 4y^4$

6. $49c^4 - 11c^2d^2 + 25d^4$

7. $16x^4 - 9x^2 + 1$

8. $100x^4 - 61x^2 + 9$

9. $225a^4b^4 - 4a^2b^2 + 4$

10. $32a^4 + 2b^4 - 56a^2b^2$

11. $a^4 + 4b^4$

12. $1 + 64x^4$

13. $x^4y^4 + 324$

Factor and check each of the following miscellaneous examples:

14. $a^4 + 2a^2x^2 + x^4$
15. $a^4 + a^2x^2 + x^4$
16. $a^4 + 4a^2x^2 + 3x^4$
17. $a^4 - 4x^4$
18. $a^4 + 4x^4$
19. $4a^4 - 13a^2x^2 + x^4$
20. $4a^4 - 4a^2x^2 + x^4$
21. $a^2 - ax - 6x^2$
22. $b^4 + a^2b^2 + a^4$
23. $a^4 + 2a^2b^2 + b^4$
24. $a^8 - x^8$
25. $a^8 + 64x^8$

26. Make up and work an example in each case in factoring treated thus far.

102. Other Methods of Factoring algebraic expressions will be treated later. Thus it will be found that

$$a^2 + b^2 = (a + b + \sqrt{2ab})\,(a^2 + b^2 - \sqrt{2ab})$$

Also, $$a^2 + b^2 = (a + \sqrt{-1}b)\,(a - \sqrt{-1}b)$$

Factoring by use of the *Factor Theorem* is treated in the Appendix (pp. 467–469).

103. General Principles in Factoring. It is important in factoring to reduce each expression to its prime factors. Therefore it is important to use the different methods of factoring in such a way as to obtain prime factors most readily.

Hence, in factoring any given expression, it is useful to

1. *Observe, first of all, whether all the terms of the expression have a common factor* (*Case I*); *if so, remove it.*

2. *Determine which other case in factoring can be used next to the best advantage.*

3. *If the expression comes under no case directly, try to discover its factors by rearranging its terms; or by adding and subtracting the same quantity; or by separating one term into two terms.*

4. *Continue the process of factoring until each factor can be resolved no further.*

EXERCISE 45

REVIEW

Factor:

1. $3x^3 - 3x$

2. $2x^3 - 8x^2y + 8xy^2$

3. $x^3 - 11x^2 + 30x$

4. $4x^3 + 5x^2y - 6xy^2$

5. $12a^2 - 2ab - 30b^2$

6. $x^4 - 1 - y^2 + 2y$

7. $40a^3 - 5$

8. $16x^4 - 40x^3y + 25x^2y^2$

9. $x^2 + 3ax - 3a - x$

10. $3x^7 - 3x$

11. $4a^4 - 5a^2 + 1$

12. $2x^8 - 32$

13. $x^2 + 4x - 45$

14. $4x^2 + 2a - a^2 - 1$

15. $5ax^9 - 5a$

16. $18x^3 - 3x^2 - 36x$

17. $x^4 + 3x^2z^2 + 4z^4$

18. $a^2x^2 - 9x^2 - a^2 + 9$

19. $110 - x - x^2$

20. $3x^2 + 13xy - 30y^2$

21. $7a - 7a^3b^4$

22. $6x^2 + 14x + 8$

23. $x^4 - (x - 2)^2$

24. $3a + 3a^4$

25. $a^3 - a^2 + 2a - 2$

26. $6x^3 - 2x - 4x^2$

27. $1 - 23z^2 + z^4$

28. $128 - 2y^3$

29. $1 - a^2 - b^2 - 2ab$

30. $21a^2 - 17a - 30$

31. $x^{12} + y^{12}$

32. $8x^3 + 729z^9$

33. $405x^4y^4 - 45x^4$

34. $a^5 - 4a^3 + 5a^2 - 20$

35. $(c + d)^3 - 1$

36. $(x - y)^2 + 2(x - y)$

37. $24x^2 + 5xy - 36y^2$

38. $x^3 - 2x^2y - 4xy^2 + 8y^3$

39. $(a^2 - 6)^2 - a^2$

40. $z^4 + z^2 + 1$

41. $(a^2 - b^2 - c^2)^2 - 4b^2c^2$

42. $21x^2 - 40xy - 21y^2$

43. $32 + n^5$

44. $5x^7 + 5xy^6$

45. $m^7 + n^7$

46. $2ax^3 + \frac{1}{4}ay^3$

47. $1 + x - x^4 - x^5$

48. $x^2 - 9 - 7(x - 3)^2$

49. $4a^4 - 37a^2 + 9$

50. $x^6 - 64$

51. $x^3 - 27 - 7(x - 3)$

52. $32x^5y - yz^{10}$

53. $(x^2 + y^2)^4 - 16x^4y^4$

54. $x^4 + x^2y^2 - y^2z^2 - z^4$

55. $ax^4 - ax - x^3y + y$

56. $4(a^2 - b^2) - 3(a + b)^2$

57. $(a - b)^3 + (x - y)^3$

58. $(a^2b - ab^2)^2$

59. $x^{12} + a^9$

60. $x^3 + y^3 + (x + y)^3$

61. $(a - b)^2 (x + y) + (a - b) (x + y)^2$

62. $(a - b)^2 + 4(a - b) (x + y) + 4(x + y)^2$

63. $a^{12} - 1$

64. $4a^2 - 9b^2 + 4a - 6b$

65. $4a^2 - 9b^2 - 1 - 6b$

66. $(x^2 - y^2)^2$

67. $(x^2 - 1)^2 + (2x + 3) (x - 1)^2$

68. $a^2 - b^4 - a^2x^3 + b^4x^3$

69. $3x^2 - 27 + ax^2 - 9a$

70. $a^3 + 3a^2b + 3ab^2 + b^3$

71. $a^3 - 3a^2x + 3ax^2 - x^3$

72. $a^2bcx - amnpx + m^2npy - abcmy$

73. $4x + 4an + x^2 - 4a^2 - n^2 + 4$

74. $2(x^3 - 8) + 7x^2 - 17x + 6$

75. $a^4 - 4b^4 + a^2 + 2b^2$

76. $(3x^3y - 3xy^3)^2$

77. $18x^2 + 52xy - 6y^2$

78. $(x + 1)^3 - x^6$

79. $(1 - 2x)^2 - x^4$

80. $ax^2 - cx + ax - c$

81. $x^4 - 79x^2 + 1$

82. $a^2 - 9 + 9b^2 - 6ab$

83. $x^6 - 4x^4 - 16x^2 + 64$

84. $(x^2 + 3)^3 - 64x^6$

85. $x^4 - 49y^2 + 9 - 6x^2$

86. $x^4y^4 - 4x^2 + 4 - y^2 - 4x^2y^2 + 4xy$

87. $a^2nx - bcm^2yz + acmxz - abmny$

88. How many of the examples in this Exercise can you work at sight?

89. Work again the odd-numbered examples on p. 88.

104. Factorial Method of Solving an Equation.

Ex. 1. Solve $x^2 + 5x - 24 = 0$.

Factoring the left-hand member, we obtain

$$(x + 8)(x - 3) = 0$$

If any factor of a product equals zero, the entire product equals zero. Hence to obtain the roots for the above equation, we may let each factor in the left-hand member equal zero and obtain the value of x from the two resulting simple equations.

Hence we have for the above equation

$$\begin{aligned} x + 8 &= 0 \\ x &= -8 \text{ Root} \end{aligned}$$

CHECK for $x = -8$

$$\begin{aligned} &x^2 + 5x - 24 \\ &= 64 - 40 - 24 \\ &= 0 \end{aligned}$$

Also $x - 3 = 0$

$$x = 3 \text{ Root}$$

CHECK for $x = 3$

$$\begin{aligned} &x^2 + 5x - 24 \\ &= 9 + 15 - 24 \\ &= 24 - 24 \\ &= 0 \end{aligned}$$

Ex. 2. Solve $x(x - 2)(3x + 4)(x + 1) = 0$.

Using the above method, we obtain

$$x = 0, 2, -\tfrac{4}{3}, -1 \text{ Roots}$$

CHECK for $x = 0$.

$$\begin{aligned} &x(x - 2)(3x + 4)(x + 1) \\ &= 0(0 - 2)(0 + 4)(0 + 1) \\ &= 0(-2)4 \times 1 = 0 \end{aligned}$$

Let the pupil apply the checks for the other values of x.

Ex. 3. Solve $x^3 - x^2 = 4x - 4$.

Transposing all terms to the left-hand member, we have

$$x^3 - x - 4x + 4 = 0$$

Hence,

$$\begin{aligned} x^2(x - 1) - 4(x - 1) &= 0 \\ (x - 1)(x^2 - 4) &= 0 \\ (x - 1)(x + 2)(x - 2) &= 0 \\ x &= 1, 2, -2 \text{ Roots} \end{aligned}$$

Let the pupil check the work.

EXERCISE 46

Solve and check each of the following:

1. $x^2 - 5x + 6 = 0$
2. $x^2 - x - 2 = 0$
3. $x^2 - 7x = -12$
4. $x^2 - x = 6$
5. $x^2 = x + 12$
6. $x^2 - 16 = 0$
7. $x^2 = 9$
8. $x(x^2 - 4) = 0$
9. $x^3 - 25x = 0$
10. $x^3 = 9x$
11. $2x^2 - 3x + 1 = 0$
12. $3x^2 - 4x = 4$
13. $x^3 - x^2 - 6x = 0$
14. $x^2 + 2x = 0$
15. $x^2 + ax = 0$
16. $x^3 - a^2x = 0$
17. $x^3 + x^2 = 4x + 4$
18. $x^3 + x^2 - 9x - 9 = 0$
19. $x^4 - 5x^2 + 4 = 0$
20. $x^3 - x^2 - 4x + 4 = 0$
21. $3(x^2 - 1) - 2(x + 1) = 0$
22. $1 + x^4 = 2x^2$
23. $y^4 - 9y^2 = 0$
24. $p^2 - 3p + 2 = 0$
25. $3m^3 - 4m + 1 = 0$
26. $z^2 - 4z + 4 = 0$
27. $y^4 - 13y^2 + 36 = 0$

Form the equation whose roots are

28. 3 and 4
29. $-5, 2$
30. $-3, -7$
31. $1, 2, -2$
32. $0, 2$
33. $2, 3, 0$

34. The square of a certain number diminished by 4 times the number equals 45. Find the number.

35. The square of a certain number increased by 6 times the number equals 40. Find the number.

36. What number plus its square equals 12?

37. The square of a certain number diminished by 9 times the number equals zero. Find the number.

38. The square of what number equals 25 times the number?

39. The cube of what number equals 25 times the number?

40. Find two consecutive numbers whose product is 72.

41. If to 3 times the square of a certain number we add 4 times the number, the result equals 4. Find the number.

42. The depth of a certain lot equals three times the front, and the area of the lot is 7500 sq. ft. Find the dimensions of the lot.

43. The temperature at which iron fuses is 2800° F., which is 332° more than 4 times the temperature at which lead fuses. Find the temperature at which lead fuses.

44. The area of Texas is 265,780 sq. mi. This is 29,240 sq. mi. less than 6 times the area of New York. Find the area of New York.

45. How many of the examples in this Exercise can you work at sight?

46. How many examples in Exercise 30 (p. 120) can you work at sight?

CHAPTER IX

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

105. Utility in the Highest Common Factor and Lowest Common Multiple. The advantages in the knowledge and use of the largest factor common to two or more expressions and of their lowest common multiple are similar to those found in arithmetic for the same principles. They aid in reducing fractions to a simple form, in adding and subtracting fractions, and in multiplying and dividing fractions. Other advantages will appear later.

Why do we now proceed to make definitions and rules?

HIGHEST COMMON FACTOR

106. A common factor of two or more algebraic expressions is an expression which divides each of the given expressions without a remainder.

The **highest common factor** of two or more algebraic expressions is the product of all their prime common factors.

Thus, the highest common factor, or H. C. F., of $4x^2$, $12x^3$, and $16x^5y$ is $4x^2$.

107. The Method of Finding the H. C. F. is to

Factor the given expressions, if necessary;

Take the H. C. F. of the numerical coefficients;

Annex the literal factors common to all of the expressions, giving to each factor the lowest exponent which it has in any expression.

Ex. 1. Find the H. C. F. of $6x^2y - 12xy^2 + 6y^3$ and $3x^2y^2 + 9xy^3 - 12y^4$.

$$\begin{aligned} 6x^2y - 12xy^2 + 6y^3 &= 6y(x - y)^2 \\ 3x^2y^2 + 9xy^3 - 12y^4 &= 3y^2(x^2 + 3xy - 4y^2) = 3y^2(x + 4y)(x - y) \\ \therefore \text{H. C. F.} &= 3y(x - y) \end{aligned}$$

Ex. 2. Find the H. C. F. of $3a(a^2b - ab^2)^2$ and $a^2(b - a)^2$.

$$\begin{aligned} 3a(a^2b - ab^2)^2 &= 3a[ab(a - b)]^2 = 3a^3b^2(a - b)^2 \\ a^2(b - a)^2 &= a^2[-(a - b)]^2 = a^2(a - b)^2 \\ \therefore \text{H. C. F.} &= a^2(a - b)^2 \end{aligned}$$

EXERCISE 47

Find the H. C. F. of

1. $4a^2b,\ 6ab^2$

2. $5x^3y,\ 15x^2y^2$

3. $24a^2x^3,\ 56a^3x^2$

4. $24xy,\ 48ax^2,\ 36x$

5. $34a^5x^3,\ 51ax^5$

6. $x^2 - 3x,\ x^2 - 9$

7. $4x^2 + 6x,\ 6x^2 + 9x$

8. $a^3 - x^3,\ a^2 - x^2$

9. $xy - y,\ x^3 - x$

10. $4a^3 + 2a^2,\ 4a^3 - a$

11. $x^2 + x,\ x^2 - 1,\ x^2 - x - 2$

12. $4a^3x - 4ax^3,\ 8a^2x^3 - 8ax^4,\ 4a^2x^2(a - x)^2$

13. $2x^3 - 2x,\ 3x^4 - 3x,\ 4x(x - 1)^2$

14. $6x^2 + 5xy - 4y^2,\ 4x^2 + 4xy - 3y^2$

15. $3x^3 - 5x^2 - 2x,\ 4x^3 - 5x^2 - 6x,\ x^3 - 4x$

16. $b - a^2b,\ 3b - a^2b - 2a^4b,\ b^2 - a^4b^2$

17. $1 - a^3,\ 1 - a^6,\ 3a + 3a^2 + 3a^3,\ 1 + a^2 + a^4$

18. Find the H. C. F. of the numerator and denominator of the fraction in Ex. 5 (p. 170).

19. Beginning with Ex. 17 (p. 170), treat each example through Ex. 22 in the same way.

Find the H. C. F. of

20. $(a^2b - ab^2)^3, \ - a^2b^2(a - b)^4$

21. $9(x^2 - xy)^3, \ 12x^2(x^2 - y^2)^2$

22. $(a + b)(x - y), \ (a - b)(y - x)$

23. $(a + b)(x - y)^2, \ (a - b)(y - x)^2$

24. $4 - x^2, \ x^2 - x - 2, \ (2 - x)^2$

25. $3a^2 - 10a + 3, \ 9a - a^3, \ (3 - a)^3$

26. $x^2(x - a)^2, \ x(a^2 - x^2)$

27. In Exs. 1 and 6, name some common factor of the two given expressions which is not their H. C. F.

28. Write two expressions whose H. C. F. is a^2x^3.

29. Write also two expressions whose H. C. F. is $3x(x - 1)$.

30. Write three expressions whose H. C. F. is $a(x - b)$.

31. How many of the examples in this Exercise can you work at sight?

LOWEST COMMON MULTIPLE

108. A common multiple of two or more algebraic expressions is an expression which will contain each of them without a remainder.

The **lowest common multiple** of two or more algebraic expressions is the expression of lowest degree which will contain them all without a remainder.

Thus, the lowest common multiple, or L. C. M., of $3a^3$, $6a^2x$, and $4ax^2$ is $12a^3x^2$.

109. The Method of Finding the L.C.M. is to

Factor the given expressions, if necessary;

Take the L. C. M. of the numerical coefficients;

Annex each literal factor that occurs in any of the given ex-

pressions, giving the factor the highest exponent which it has in any one expression.

Ex. 1. Find the L. C. M. of $3x^4 - 9x^3$, $x^3 - 9x$, and $x^2 - 6x + 9$.

$$\begin{aligned} 3x^4 - 9x^3 &= 3x^3(x-3) \\ x^3 - 9x &= x(x+3)(x-3) \\ x^2 - 6x + 9 &= (x-3)^2 \\ \therefore \text{L. C. M.} &= 3x^3(x+3)(x-3)^2 \end{aligned}$$

Ex. 2. Find the L. C. M. of $(a^2b - ab^2)^3$, $2ab(b-a)^2$, and $a^2(a^2-b^2)^2$.

$$\begin{aligned} (a^2b - ab^2)^3 &= [ab(a-b)]^3 = a^3b^3(a-b)^3 \\ 2ab(b-a)^2 &= 2ab[-(a-b)]^2 = 2ab(a-b)^2 \\ a^2(a^2-b^2)^2 &= a^2(a+b)^2(a-b)^2 \\ \therefore \text{L. C. M.} &= 2a^3b^3(a+b)^2(a-b)^3 \; \textit{Ans.} \end{aligned}$$

EXERCISE 48

Find the L. C. M. of

1. $3a^2b$, $2ab^2$
2. $12a^2x^2$, $9a^2y^2$
3. $2ac$, $3bc$, $4ab$
4. $3a^2b$, $4ac^2$, $6b^2c$
5. $42x^3y^2$, $28y^3z^2$
6. $12a^2b$, $16ab^2$, $24a^2b^2$
7. $2x(x+1)$, $x^2 - 1$
8. $3a^2 + 3ab$, $2ab + 2b^2$
9. $7x^2$, $2x^2 - 6x$
10. $x^3 - 1$, $x^2 - 1$
11. $x^2 - y^2$, $x^2 - 3xy + 2y^2$
12. $3x^3 - 3x$, $6x^2 - 12x + 6$
13. $5ax^2(x-y)^2$, $3bxy(x^2 - y^2)$
14. $x^3 - 3x^2 - 40x$, $x^2 - 9x + 8$
15. $a^2 - b^2$, $a^3 - b^3$, $a^3 + b^3$
16. $6x^2 + 6x$, $2x^3 - 2x^2$, $3x^2 - 3$
17. $4a^2b + 4ab^2$, $6a^2b - 6ab^2$, $3a^2 - 3b^2$
18. $2x^2 + x - 1$, $4x^2 - 1$, $2x^2 + 3x + 1$
19. $3x^3 - 3$, $6x^2 - 12x + 6$, $2x^3 + 2x^2 + 2x$

20. $12x^3 - 2x^2 - 140x,\ 18x^2 + 6x - 180,\ 6x^3 - 39x^2 + 63x$

21. $1 - x + x^2 - x^3,\ 1 + x + x^2 + x^3,\ 2x - 2x^3$

22. $(x - 1)^3,\ 7xy^3(x^2 - 1)^2,\ 14x^5y(x + 1)^3$

23. $18x^3 - 12x^2 + 2x,\ 27x^5 - 3x^3,\ 18x^3 - 24x^2 + 6x$

24. $(x - 1)(x + 3)^2,\ (x + 1)^2(x - 3),\ (x^2 - 1)^2,\ x^2 - 9$

25. Find the L. C. M. of the denominators of the fractions in Ex. 18 (p. 181).

26. Find the L. C. M. also of the denominators of the fractions in each example from Ex. 21 to Ex. 28 (inclusive), p. 181.

Find the L. C. M. of

27. $(a^2b - ab^2)^4,\ a^3b^3(a + b)^2$

28. $(abc - bcd)^3,\ (3a^2c - 3acd)^2,\ 6a^2c^2 - 6a^2d^2$

29. $(a^2b - ab^2)^2,\ (a^2 - ab)^2,\ (a^3 + a^2b)^2$

30. $9(x^2 - xy)^3,\ 12(x^2 - y^2)^2,\ 18(x^3 + x^2y)^2$

31. $a - b,\ b - a$

32. $9(a - b)^2,\ 12(b - a)^2$

33. $(a + b)(x - y),\ (a - b)(y - x)$

34. $(a + b)(x - y)^2,\ (a - b)(y - x)^3$

35. $4 - x^2,\ x^2 - x - 2,\ (2 - x)^2$

36. $x^2(x - a)^2,\ x(a^2 - x^2)$.

37. Find two consecutive numbers the difference of whose squares is 5.

38. Make up and work an example similar to Ex. 37.

39. The reclaimable swamp land in the United States and the land that is capable of irrigation equal 178,000,000 acres all together. If the irrigable land exceeds the swamp land by 22,000,000 acres, how many acres of each of these kinds of land are there?

40. The distance from New York to Havana is 1410 mi. If a steamer leaving New York travels at the average rate of 260 mi. per day, and one leaving Havana at the same time travels at the average rate of 280 mi. per day, how many days and hours will elapse before the two steamers meet?

41. The distance of the sun from the earth is 92,800,000 mi. This distance exceeds 107 times the diameter of the sun by 95,200 mi. Find the diameter of the sun.

42. A man bequeathed $20,000 to his wife, daughter, and son. To his daughter he left $2000 more than to his son, and to his wife three times as much as to his son. How much did he leave to each?

43. The distance of the moon from the earth is 238,850 mi. This exceeds 110 times the moon's diameter by 1030 mi. Find the diameter of the moon.

44. If 10 m. exceeds 10 yd. by 33.7 in., how many inches are there in a meter?

45. Write a common multiple of the expressions in Ex. 1, which is not their L. C. M.

46. Write a common multiple of the expressions in Ex. 10 which is not their L. C. M.

47. Write two expressions whose L. C. M. is $24a^2b^3c^2$.

48. Write two expressions whose L. C. M. is $12x^3(x-2)^2(x-1)$.

49. Make up and work an example similar to Ex. 27. To Ex. 31. To Ex. 47.

50. How many of the examples in this Exercise can you work at sight?

51. How many examples in Exercise 35 (p. 131) can you work at sight?

CHAPTER X

FRACTIONS

110. Utility of Fractions. In algebra, as in arithmetic, fractions are useful in indicating new units, and in indicating quotients and thus opening the way to save labor by cancellation.

In algebra fractions also have other uses besides those which appear in arithmetic. Thus, in algebra, a fraction is often useful in expressing a general formula.

Ex. If an automobile goes a miles in b hours, how far would it go in c hours at the same rate?

$\frac{a}{b}$ = no. of miles the automobile travels in 1 hour

$\frac{ac}{b}$ = no. of miles the automobile travels in c hours

Why do we now proceed to make definitions and rules?

111. A Fraction is the indicated quotient of two algebraic expressions. This quotient is usually indicated in the form $\frac{a}{b}$.

The fraction $\frac{a}{b}$ is read "a divided by b," or, for brevity, "a over b."

Note that the dividing line of a fraction takes the place of a parenthesis and is in effect a *vinculum*.

Another method of writing the preceding fraction is a/b. This is called the *solidus notation*. It is convenient in printing mathematical expressions, and is much used in European mathematical literature.

$\frac{x+1}{3x-5}$ written in the solidus notation would be $(x + 1)/(3x - 5)$

The *numerator* of a fraction is the dividend and the *denominator* is the divisor of the indicated quotient.

Terms of a fraction is a general name for both numerator and denominator.

EXERCISE 49

1. If three boys weigh a, b, c pounds respectively, what is their average weight?

2. If four boys can run the quarter mile in p, q, r, s seconds respectively, what is their average time?

3. How many acres are there in a field a feet long and b feet wide?

4. How many acres are there in a field c rd. × d rd.? In one f yd. × e ft.? p ft. × q rd.?

5. If sugar is worth a cents a pound, how many pounds can be obtained in exchange for b pounds of butter worth c cents a pound?

6. If coal is worth c dollars a ton, how many tons of coal can be obtained in exchange for p tons of hay worth b dollars a ton?

7. Make up and work a similar example concerning c calves, worth a dollars each, exchanged for chairs worth d dollars each.

8. If coal is worth c dollars a ton, how many tons can be obtained in exchange for f bushels of wheat worth h cents a bushel and for w bushels of corn worth y cents a bushel?

9. Who first used the letters a, b, c to represent known numbers? (See p. 456.) Tell all you can about this man.

10. Before the use of a, b, c, what other symbols were used to represent known numbers? Discuss the relative advantages in these different sets of symbols.

11. As a notation, in what respects is a/b superior to $a \div b$? To $\frac{a}{b}$? In what respects is it inferior to each of these?

12. How many examples in Exercise 45 (p. 155) can you now work at sight?

112. **An integral expression** is one which does not contain a fraction; as $3x^2 - 2y$.

An expression like $5x^2 + \frac{3}{2}x + \frac{1}{4}$ in which fractions occur only in the numerical coefficients is sometimes regarded as an integral expression.

A mixed expression is one which is part integral, part fractional.

$$\text{Thus, } 3x^2 + x - 5 + \frac{x+1}{3x^2 - 2}$$

113. Sign of a Fraction. A fraction has its own sign, which is distinct from the sign of both numerator and denominator. It is written to the left of the dividing line of the fraction.

The sign of $-\frac{a}{b}$ is $-$, and the sign of $\frac{-a}{b}$ is $+$ understood.

GENERAL PRINCIPLES

114. A. *If the numerator and the denominator of a fraction are both multiplied or divided by the same quantity, the value of the fraction is not changed.*

For if a dividend is denoted by D, its divisor by d, and the quotient by Q

$$\frac{D}{d} = Q, \text{ and } D = d \times Q$$

If m denotes any multiplier, $D \times m = d \times m \times Q$, or

$$\frac{D \times m}{d \times m} = Q \qquad \text{(Art. 15, p. 18)}$$

Also if m denotes any divisor except zero,

$$D \div m = d \div m \times Q, \text{ or } \frac{D \div m}{d \div m} = Q \qquad \text{(Art. 15)}$$

115. B. **Law of Signs.** By the laws of signs for multiplication and division (see Arts. 50, 62, pp. 59, 77),

$$\frac{a}{b} = \frac{-a}{-b}, \quad -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}, \quad \frac{a}{bc} = -\frac{a}{-b \times c} = \frac{a}{-b \times -c}$$

$$\frac{x+y}{y-x} = \frac{x+y}{-(x-y)} = -\frac{x+y}{x-y}$$

Or, in general,

The signs of any **even** *number of factors of the numerator and denominator of a fraction may be changed without changing the sign of the fraction.*

But if the signs of an **odd** *number of factors are changed, the sign of the fraction must be changed.*

TRANSFORMATIONS OF FRACTIONS

I. TO REDUCE A FRACTION TO ITS LOWEST TERMS

116. A Fraction in its Lowest Terms is a fraction whose numerator and denominator have no common factor.

To reduce a fraction to its lowest terms, as in arithmetic,

Resolve the numerator and the denominator into their prime factors, and cancel the factors common to both.

Ex. 1. Reduce $\frac{36a^3x^2}{48a^2x^3y^2}$ to its lowest terms.

Divide both numerator and denominator by $12a^2x^2$ (see Art. 114).

$$\therefore \frac{36a^3x^2}{48a^2x^3y^2} = \frac{3a}{4xy^2} \; Ans.$$

Ex. 2. $\frac{9ab - 12b^2}{12a^2 - 16ab} = \frac{3b(3a - 4b)}{4a(3a - 4b)} = \frac{3b}{4a}$ *Ans.*

Notice particularly that in reducing a fraction to its lowest terms it is allowable to cancel a *factor* which is common to both denominator and numerator, but it is not allowable to cancel a *term* which is common unless this term is a factor.

Thus, $\frac{ab}{ac}$ reduces to $\frac{b}{c}$;

but in $\frac{a+x}{a+y}$, a of the numerator will not cancel a of the denominator.

This is a principle very frequently violated by beginners.

EXERCISE 50

Reduce each of the following to its simplest form:

1. $\frac{27}{36}$
2. $\frac{108}{144}$
3. $\frac{72}{150}$
4. $\frac{8a^3x^4}{12a^2x^5}$
5. $\frac{12x^4yz^5}{15x^3y^2z^5}$
6. $\frac{3a^2x}{6a^2 - 9a^3x}$
7. $\frac{72x^2y^3z^4}{96xy^5z^3}$
8. $\frac{3a^3 - 6a^2b^2}{4a^2b^2 - 8ab^3}$
9. $\frac{2a}{4a^2 - 2a}$
10. $\frac{3x - 6y}{6ax - 12ay}$
11. $\frac{4x + 4y}{4ax + 4ay}$
12. $\frac{x^2 - y^2}{(x + y)^2}$
13. $\frac{12a^2x^2 - 8a^2xy}{18ax^3 - 12ax^2y}$
14. $\frac{8(x^2 - 1)}{12x - 12}$
15. $\frac{45(x - y)^2}{18(x - y)^3}$
16. $\frac{a^2b + ab^2}{2a^2b - 2ab^2}$
17. $\frac{6xy}{9x^2y - 12xy^2}$
18. $\frac{6a^2b^2 + 12ab^3}{9a^3b + 18a^2b^2}$
19. $\frac{2x^2 - 3xy}{4x^3 - 9xy^2}$
20. $\frac{49x^2 - 64y^2}{14x^3 - 16x^2y}$
21. $\frac{x^3 - 27}{x^2 - 6x + 9}$
22. $\frac{(x - y)^2\,(x + y)^3}{(x^2 - y^2)^3}$
23. $\frac{2x^2 - 8y^2}{4x^2 - 2xy - 12y^2}$
24. $\frac{6x^2 - xy - 2y^2}{6x^2 - 7xy + 2y^2}$
25. $\frac{(a + b)^2 - c^2}{a^2 - (b + c)^2}$

26. $\dfrac{1-(a-x)^2}{x^2-(a-1)^2}$

29. $\dfrac{x^4-9x^2}{x^4-x^3-6x^2}$

27. $\dfrac{12x^2-2ax-24a^2}{4x^2-2ax-6a^2}$

30. $\dfrac{x^6-y^6}{x^4+x^2y^2+y^4}$

28. $\dfrac{x^3-8}{x^2y^2+2xy^2+4y^2}$

31. $\dfrac{ax-bx-ay+by}{a^2-b^2}$

32. $\dfrac{x^2-z^2-4-2xy-4z+y^2}{z^2-x^2-4-2yz-4x+y^2}$

33. What is the correct value of the fraction $\dfrac{1+4}{6+4}$? If the 4's are struck out, what does the value of the above fraction become? Is it allowable, therefore, to strike out the 4's in the above fraction?

34. Make up and work an example similar to Ex. 33.

35. Which of the following fractions can be simplified by striking out the 4's?

$$\frac{1+4}{11+4} \qquad \frac{x+4}{y+4} \qquad \frac{4x}{4(1+y)} \qquad \frac{3\times 4}{4+11} \qquad \frac{4a}{x+4} \qquad \frac{4a}{4x}$$

36. Make up and work an example similar to Ex. 35, involving 3's.

37. Which of the following can be simplified by striking out the b^2's?

$$\frac{b^2+x}{b^2+y} \qquad \frac{b^2x}{b^2y} \qquad \frac{b^2-4}{3b^2+4} \qquad \frac{a^2b^2}{a^2+b^2} \qquad \frac{a^2b^2}{b^2x^4}$$

38. Which of the following can be simplified by striking out $a+b$ in both numerator and denominator?

$$\frac{a+b}{a+b+c} \qquad \frac{5(a+b)}{3(a+b)+c} \qquad \frac{3x(a+b)}{4y(a+b)} \qquad \frac{3(a+b)}{5(a+b)+c}$$

39. Make up and work an example similar to Ex. 38 concerning the striking out of x. Of $(p+q)^2$.

40. Why is it allowable to subtract 4 from each member of the equation $x + 4 = a + 4$ and not from each term of the fraction $\frac{x+4}{a+4}$?

41. How many of the examples in this Exercise can you work at sight?

EXERCISE 51

1. Reduce $\frac{(x-2)^2}{4-x^2}$ to its lowest terms.

$$\frac{(x-2)^2}{4-x^2} = \frac{(x-2)(x-2)}{(2+x)(2-x)} = \frac{(2-x)(2-x)}{(2+x)(2-x)} = \frac{2-x}{2+x}$$

CHECK. Let $x = 1$, then, $\frac{(x-2)^2}{4-x^2} = \frac{(-1)^2}{4-1} = \frac{1}{3}$

Also, $\frac{2-x}{2+x} = \frac{2-1}{2+1} = \frac{1}{3}$

Reduce to simplest form and check the work:

2. $\frac{(a-b)^2}{b^2-a^2}$

3. $\frac{(2x-y)^2}{y^2-4x^2}$

4. $\frac{9-m^2}{m^2-7m+12}$

5. $\frac{9-x^2}{(x-3)^2}$

6. $\frac{2-y}{y^2-4}$

7. $\frac{a-b}{b-a}$

8. $\frac{a+b-c}{c^2-(a+b)^2}$

9. $\frac{6y-3x}{12ay-6ax}$

10. $\frac{6a^2b-3a^3}{4a^2b^2-8ab^3}$

11. $\frac{x^3-27}{9-6x+x^2}$

12. $\frac{4-(a+b)^2}{(a-2)^2-b^2}$

13. $\frac{ax-bx-ay+by}{b^2-a^2}$

Without changing the value of the fraction

14. Change each of the following so that the denominator of the fraction shall be $a - b$.

$$\frac{-3}{b-a} \qquad \frac{3}{b-a} \qquad \frac{x}{b-a} \qquad \frac{x-y}{b-a} \qquad \frac{-3x}{b-a} \qquad \frac{2a-3b}{b-a}$$

15. Change each of the following so that the denominator shall be $(x - y)(x - y)$.

$$\frac{3}{(y-x)(y-x)} \qquad \frac{-4}{(y-x)(x-y)} \qquad \frac{a-b}{(y-x)^2}$$

16. Show that $\dfrac{1}{(y-x)(z-y)}$ equals $\dfrac{1}{(x-y)(y-z)}$

17. By changing signs of factors, write each of the following in three different ways:

$$\frac{5}{a-b} \qquad \frac{a-b}{x-y} - \frac{a-b}{c-d} \qquad \frac{a-b}{(x-y)(y-z)} \qquad \frac{(b-a)(c-d)}{(y-x)(y-z)}$$

Solve the following equations, after reducing the fraction in each equation to its simplest form:

18. $\dfrac{x^2-1}{x-1} + 2x = 5$

19. $\dfrac{x^3-1}{x-1} - x^2 = 7$

20. $\dfrac{x^3+27}{x+3} = 7 + x^2$

21. $\dfrac{x^3-3x^2}{x^2} = 5 - x$

22. $\dfrac{ax-3a+bx-3b}{a+b} = 17$

23. $\dfrac{ax-5a-bx+5b}{a-b} = 15 - 3x$

24. $\dfrac{x^3-8}{x-2} - \dfrac{x^3+8}{x+2} = 20$

25. How many of the examples in this Exercise can you work at sight?

II. TO REDUCE AN IMPROPER FRACTION TO AN INTEGRAL OR MIXED QUANTITY

117. An Improper Fraction is one in which the degree of the numerator equals or exceeds the degree of the denominator.

Since a fraction is an indicated division, to reduce an improper fraction to an integral or mixed expression,

Divide the numerator by the denominator;

If there is a remainder, write it over the denominator, and annex the result to the quotient with the proper sign.

Ex. Reduce $\frac{x^3 + 4x^2 - 5}{x^2 + x + 2}$.

$$\begin{array}{rrrr|l}
x^3 & + 4x^2 & & - 5 & x^2 + x + 2 \\
x^3 & + x^2 & + 2x & & \overline{\quad x + 3} \\
\hline
& 3x^2 & - 2x & - 5 & \\
& 3x^2 & + 3x & + 6 & \\
\cline{2-4}
& & - 5x & - 11 &
\end{array}$$

$$\therefore \frac{x^3 + 4x^2 - 5}{x^2 + x + 2} = x + 3 - \frac{5x + 11}{x^2 + x + 2} \quad \textit{Ans.}$$

When the remainder is made the numerator of a fraction with the minus sign before it, as in this example, the signs of terms of the remainder must be changed, since the vinculum is in effect a parenthesis (see Art. 41, p. 50).

EXERCISE 52

Reduce each of the following to a mixed quantity:

1. $\frac{32}{5}$

2. $\frac{121}{9}$

3. $\frac{181}{17}$

4. $\frac{x^2 - 2x + 3}{x}$

5. $\frac{4x^3 + 6x - 5}{2x}$

6. $\frac{10a^3x^3 + 5ax - 7 - a}{5ax}$

7. $\frac{x^3 - 3x^2 + x - 1}{x + 1}$

8. $\dfrac{x^2 + 3xy - 2y^2 - 1}{x + y}$

9. $\dfrac{3x^4 - 13x - 28}{x^2 - 3}$

10. $\dfrac{x^3 - x^2 - x + 2 - a}{x - 1}$

11. $\dfrac{x^4 + 1}{x^2 + x - 1}$

12. $\dfrac{x^5}{x^2 - x - 1}$

13. $\dfrac{2x^4 + 7}{x^2 + x + 1}$

14. $\dfrac{x^4 + x^2 - x - 1}{x^2 + 2}$

15. $\dfrac{9a^3}{3a^2 - 2b}$

16. $\dfrac{x^3 + x^2 - 4x + 7}{x + 3}$

17. $\dfrac{2a^2}{a + b}$

18. $\dfrac{x^5 - x^4 + x^3 - 2x}{x^3 + 1}$

19. $\dfrac{1}{1 + x}$ (To three terms.)

20. $\dfrac{1}{1 + x - x^2}$

21. $\dfrac{8}{2 + x - x^2}$

22. Make up an improper fraction with a monomial denominator and reduce it to a mixed number.

23. Make up an improper fraction with a binomial denominator and reduce it to a mixed number.

24. How many examples in Exercise 1 (p. 8) can you now work at sight?

III. To Reduce a Mixed Expression to a Fraction

118. To Reduce a Mixed Expression to a fraction, it is necessary simply to reverse the process of Art. 117. Hence,

Multiply the integral expression by the denominator of the fraction, and add the numerator to the result, changing the signs of the terms of the numerator if the fraction is preceded by the minus sign;

Write the denominator under the result.

Ex. $$x+y-\frac{x^2+y^2}{x-y}$$
$$=\frac{(x+y)(x-y)-(x^2+y^2)}{x-y}$$
$$=\frac{x^2-y^2-x^2-y^2}{x-y}=\frac{-2y^2}{x-y}$$
$$=\frac{2y^2}{y-x}\ \textit{Ans.}$$

EXERCISE 53

Reduce to a fraction:

1. $3\frac{1}{7}$
2. $12\frac{2}{9}$
3. $13\frac{5}{12}$
4. $a-1+\dfrac{1}{a}$
5. $x+1+\dfrac{1}{x-1}$
6. $x^2+x-1-\dfrac{1}{x-1}$
7. $4x-2-\dfrac{y-2}{2x+1}$
8. $a-b+\dfrac{2b^2}{a+2b}$
9. $x-1-\dfrac{x-1}{x^2+x+1}$
10. $a-x+1-\dfrac{x-1}{a+x}$
11. $\dfrac{3a-1}{a-2}+a-1$
12. $x-a-\dfrac{ay-a^2}{x+a}+y$
13. $1-\dfrac{b^2-c^2+a^2}{2bc}$
14. $6a+\dfrac{(2-3a)^2}{4}$
15. $\dfrac{(a+b)^2}{4}-ab$
16. $\dfrac{(2ab-1)^2}{4}+2ab$
17. $1-\left(x-x^2+\dfrac{1}{1+x}\right)$
18. $x^2-\left[x-\left(1-\dfrac{2}{x+1}\right)\right]$
19. $x^3-\left\{-x^2-\left[x+1-\dfrac{1}{1-x}\right]\right\}$

20. The distance from New York to Chicago is 912 mi., which is 100 mi. more than one fourth of the distance from New York to San Francisco. Find the latter distance.

21. A running horse with a rider has gone 1 mi. in 1 min. $35\frac{1}{2}$ sec., which is $13\frac{1}{2}$ sec. more than three times the time in which an automobile has gone one mile. Find the latter time.

22. Make up two mixed numbers of your own and reduce them to improper fractions.

23. How many of the examples in this Exercise can you work at sight?

IV. TO REDUCE FRACTIONS TO EQUIVALENT FRACTIONS OF THE LOWEST COMMON DENOMINATOR

119. To Reduce Fractions to their lowest common denominator, as in arithmetic, we

Find the lowest common multiple of the denominators of the given fractions;

Divide this common multiple by the denominator of each fraction;

Multiply each quotient by the corresponding numerator; the results will form the new numerators;

Write the lowest common denominator under each new numerator.

Ex. Reduce $\frac{2}{3ax}$, $\frac{3}{4a^2x}$, and $\frac{5}{6ax^2}$ to equivalent fractions having the lowest common denominator.

The L. C. D. is $12a^2x^2$.

Dividing this by each of the denominators, we get the quotients $4ax$, $3x$, and $2a$.

Multiplying each of these quotients by the corresponding numerator and setting the results over the common denominator, we obtain

$$\frac{8ax}{12a^2x^2}, \frac{9x}{12a^2x^2}, \frac{10a}{12a^2x^2} \text{ } Ans.$$

EXERCISE 54

Reduce the following to equivalent fractions having the lowest common denominator:

1. $\frac{5}{8}, \frac{7}{12}$

2. $\frac{3}{5}, \frac{4}{15}, \frac{9}{20}$

3. $\frac{2x}{9}, \frac{5x}{6}$

4. $\frac{12a}{5b}, \frac{7}{10}, \frac{a}{b}$

5. $\frac{1}{2ab^2}, \frac{2}{a^2b}, \frac{1}{ab}$

6. $\frac{2}{3a^2}, \frac{3}{4ax}, 2a, \frac{1}{x}$

7. $\frac{ac}{bd}, \frac{ab}{cd}, \frac{bc}{ad}, \frac{ad}{bc}$

8. $\frac{1}{a^2 - a}, 2, \frac{3}{a - 1}$

9. $\frac{x}{1 + x}, 1, \frac{1}{x}, \frac{1}{x + x^2}$

10. $\frac{x}{x^2 - 1}, \frac{1}{x^3 - 1}$

11. $\frac{1}{4x^2 - 9}, \frac{1}{2x + 3}, \frac{1}{x}$

12. $\frac{1 + x}{2 - 2x}, 7, \frac{1 - x}{3 + 3x}$

13. $\frac{3}{x^3 - 1}, \frac{4}{x^2 + x + 1}, 4$

14. $\frac{3}{a^2b + ab^2}, \frac{4}{a^2b - ab^2}$

15. $\frac{1}{3x - 6}, \frac{5}{2x + 4}, \frac{3}{x^2 - 4}$

16. $\frac{2}{x - x^3}, \frac{x}{3 + 3x}, \frac{x}{2 - 2x}$

PROCESSES WITH FRACTIONS

I. ADDITION AND SUBTRACTION OF FRACTIONS

120. The Method of Adding or Subtracting Fractions, as in arithmetic, is to

Reduce the fractions to their lowest common denominator;

Add their numerators, changing the signs of the numerator of any fraction preceded by the minus sign;

Set the sum over the common denominator;

Reduce the result to its lowest terms.

Ex. 1. $$\frac{a}{a-1} - a + \frac{1}{a^2 - a} + \frac{1}{a}$$

$$= \frac{a}{a-1} - \frac{a}{1} + \frac{1}{a^2-a} + \frac{1}{a}$$

$$= \frac{a^2 - a^3 + a^2 + 1 + a - 1}{a(a-1)}$$

$$= \frac{-a^3 + 2a^2 + a}{a(a-1)} = \frac{-a^2 + 2a + 1}{a-1} \quad \textit{Ans.}$$

Ex. 2. Simplify $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1} - \dfrac{x}{1-x}$.

The factors of $x^2 - 1$ are $x + 1$ and $x - 1$. Hence, if the sign of the denominator, $1 - x$, is changed, it will become $x - 1$, and be a factor of $x^2 - 1$. But by Art. 115 (p. 169), if the sign of $1 - x$ is changed, the sign of the fraction in which it occurs must also be changed. Hence, we have

$$\frac{x^2}{x^2-1} + \frac{x}{x+1} + \frac{x}{x-1} = \frac{x^2 + x^2 - x + x^2 + x}{x^2-1} = \frac{3x^2}{x^2-1} \quad \textit{Ans.}$$

Where the differences of three letters occur as factors in the various denominators, it is useful to have some standard order for the letters in the factors. It is customary to reduce the factors so that the alphabetical order of the letters is preserved in each factor, except that the last letter is followed by the first. This is called the *cyclic order*.

Thus, $a - b$, $b - c$, $c - a$ are written in the cyclic order.

Ex. 3. Simplify

$$\frac{1}{(a-b)(c-a)}+\frac{1}{(a-b)(c-b)}+\frac{1}{(c-b)(a-c)}.$$

Changing $c-b$ to $b-c$, and $a-c$ to $c-a$ where they occur, we obtain

$$\frac{1}{(a-b)(c-a)}-\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}$$

$$=\frac{b-c-c+a+a-b}{(a-b)(b-c)(c-a)}$$

$$=\frac{2a-2c}{(a-b)(b-c)(c-a)}=\frac{-2}{(a-b)(b-c)}\ \textit{Ans.}$$

EXERCISE 55

Find the value of

1. $\dfrac{3}{2x}+\dfrac{2}{x}-\dfrac{1}{3x}$

2. $\dfrac{2}{3a}-\dfrac{3}{4ax}+\dfrac{1}{x}$

3. $\dfrac{5}{2ac}-\dfrac{2}{3ab}-\dfrac{1}{bc}$

4. $\dfrac{a+2b}{2ab}-\dfrac{6a-1}{6a^2}$

5. $\dfrac{2a^2x+3}{4ax^2}+1-\dfrac{3a+x}{6x}$

6. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$

7. $x-\dfrac{3x+1}{8}+\dfrac{1-3x}{6}$

8. $\dfrac{a+1}{2}-\dfrac{a-1}{2}$

9. $\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}$

10. $\dfrac{x+1}{x}-3+\dfrac{7-3x}{3x^2}$

11. $\dfrac{3a-4b}{2}-\dfrac{2a-b-c}{3}+\dfrac{15a-4c}{12}$

12. $\dfrac{2x^2y-3z}{3x^2y}-\dfrac{xz^2-y^2z}{2xy^2}+\dfrac{y-3xz^2}{6x^2z}-\dfrac{2}{3}$

13. $\dfrac{x+1}{x-2}+\dfrac{1-x}{x+2}$

14. $\dfrac{m+1}{(m-1)^2}+\dfrac{2m}{m^2-1}$

15. $\dfrac{(a+b)^2}{4(a-b)^2} - \dfrac{2b}{a-b}$

16. $\dfrac{3x}{x+2} - \dfrac{2x}{x-2} + \dfrac{10x}{x^2-4}$

17. $\dfrac{1}{x^2+x} - 2 + \dfrac{2x^2}{x^2-x}$

18. $\dfrac{1}{3x-3} - \dfrac{1}{2x+2} + \dfrac{x-5}{6x^2-6}$

19. $\dfrac{2}{2x-1} + \dfrac{3}{4x+2} - \dfrac{7x}{4x^2-1}$

20. $\dfrac{x}{x^2-1} + 2 - \dfrac{x-1}{x+1} - \dfrac{x-2}{x-1}$

21. $\dfrac{3}{x+1} - \dfrac{4}{x+2} + \dfrac{2}{x+3}$

22. $\dfrac{x+2}{2x^2+x-1} - \dfrac{x-3}{4x^2-1} + \dfrac{2x+5}{2x^2+3x+1}$

23. $\dfrac{b}{a+b} - \dfrac{ab}{(a+b)^2} - \dfrac{ab^2}{(a+b)^3}$

24. $\dfrac{2xy}{x^2-y^2} + \dfrac{3y}{2x} + \dfrac{3x}{2y} - \dfrac{3x^2-3y^2}{2xy}$

25. $\dfrac{3x-y}{x+2y} + \dfrac{14xy}{x^2-4y^2} - \dfrac{3x+y}{x-2y}$

26. $1 - \dfrac{2}{x-1} + x - \dfrac{3x-1}{x+1} - \dfrac{2x-5}{2}$

27. $\dfrac{2}{x+4} - \dfrac{x-3}{x^2-4x+16} - \dfrac{x^2}{x^3+64}$

28. $\dfrac{5x}{2(x-3)^2} - \dfrac{7}{3x+9} - \dfrac{26}{4x^2-36}$

Reduce each of the following fractions to its lowest terms and collect:

29. $\dfrac{x+3}{x^2-9} - \dfrac{1}{x-4}$

30. $\dfrac{x^2+x}{x^2-1} + 1 - \dfrac{x^2+x}{(x+1)^2}$

31. $\frac{4b^2}{a^2-b^2}-\frac{(a-b)^2}{a^2-b^2}+2-\frac{3a+3b}{3a-3b}$

32. $\frac{3x}{x^2-1}+\frac{4}{1-x}+\frac{1}{1+x}$

33. $\frac{2a}{a^2-b^2}+\frac{1}{a+b}+\frac{2}{b-a}$

34. $\frac{3xy}{x^2-4y^2}-\frac{y-x}{2y+x}+\frac{y+x}{2y-x}$

35. $\frac{1}{x-1}+\frac{1}{1+x}+\frac{2x}{1-x^2}$

36. $\frac{x^2+y^2}{x^2-y^2}-\frac{x}{x+y}+\frac{y}{y-x}$

37. $\frac{3}{8-8a}+\frac{5}{4a+4}-\frac{7a}{8a^2-8}$

38 $\frac{3}{x}+\frac{2}{x-1}+\frac{5x}{1-x^2}-\frac{1}{x+1}-\frac{3}{x+x^2}$

39. $\frac{1}{(x-2)(3-x)}-\frac{1}{10-7x+x^2}-\frac{1}{(5-x)(x-3)}$

40. $\frac{2}{(a-3)(b-2)}-\frac{3}{(a-2)(2-b)}+\frac{4}{(a-2)(3-a)}$

$+\frac{5}{(a-3)(2-b)}$

41. $\frac{2b+a}{x+a}-\frac{2b-a}{a-x}-\frac{4bx-2a^2}{x^2-a^2}$

42. $\frac{x+1}{6x-6}-\frac{2x-1}{12x+12}+\frac{2}{3-3x^2}-\frac{7}{12x}$

43. $\frac{x^2-x-6}{x^2+5x+6}-\frac{x^2+4x+3}{x^2-4x+3}-\frac{15x}{9-x^2}$

44. $\frac{a^2+2ab+b^2}{a^2-b^2}-\frac{4a^2-b^2}{2a^2-3ab-2b^2}+\frac{a^2-2ab+3b^2}{a^2-3ab+2b^2}$

45. $\frac{1-x^2}{9-x^2}+\frac{x^2-9}{3(x+3)^2}-\frac{x^2-4x+3}{5(x-3)^2}-\frac{2x}{5x^2-45}$

46. $\frac{3x+2}{x^2-5x+6}+\frac{x}{8x-x^2-15}-\frac{4-x}{7x-x^2-10}$

47. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$

48. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$

49. $\frac{y+z}{(x-y)(x-z)}+\frac{z+x}{(y-z)(y-x)}+\frac{x+y}{(z-x)(z-y)}$

50. $\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-z)(y-x)}+\frac{xy}{(z-x)(z-y)}$

51. $\frac{1+l}{(l-m)(l-n)}+\frac{1+m}{(m-n)(m-l)}+\frac{1+n}{(n-l)(n-m)}$

52. $\frac{1}{x}-\left\{\frac{x}{x+1}-\left[\frac{1-x}{x^2-x+1}-\frac{1}{x+1}\right]-1\right\}-\frac{1}{x^3+1}$

53. Make up and add three fractions with monomial denominators.

54. Also three with binomial denominators.

55. How many examples in Exercise 2 (p. 13) can you now work at sight?

II. MULTIPLICATION OF FRACTIONS

121. The Method of Finding the Product of two or more fractions, as in arithmetic, is to

Multiply the numerators together for a new numerator, and multiply the denominators together for a new denominator, canceling factors that are common to the two products.

This method reduces the multiplication of fractions to the multiplication of integral expressions, and enables us to use again our knowledge of the latter process.

Ex. $\frac{x+y}{x} \times \frac{x^2-y^2}{x^3+xy^2} \times \frac{4x^2}{(x+y)^2}$

$$= \frac{x+y}{x} \times \frac{(x+y)(x-y)}{x(x^2+y^2)} \times \frac{4x^2}{(x+y)(x+y)}$$

$$= \frac{4(x-y)}{x^2+y^2} \ \textit{Ans.}$$

II. Division of Fractions

122. The Method of Dividing one fraction by another is the same as in arithmetic. For

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times d} \div \frac{b \times c}{b \times d} \text{ (see Art. 114, p. 168)}$$

$$= \frac{a \times d}{b \times c} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

Hence, to divide one fraction by another,

Invert the divisor and proceed as in multiplication.

Ex. $\frac{x^3-1}{x(x+1)} \times \frac{(x^2-1)^2}{x^2+x+1} \div \frac{(x-1)^3}{(x+1)^2}$

$$= \frac{(x-1)(x^2+x+1)}{x(x+1)} \times \frac{(x^2-1)(x^2-1)}{x^2+x+1} \times \frac{(x+1)^2}{(x-1)^3}$$

$$= \frac{(x+1)^3}{x} \ \textit{Ans.}$$

The **reciprocal** of a number is the result obtained by dividing unity by the given number.

Thus, the reciprocal of 2 is $1 \div 2$ or $\frac{1}{2}$; of x is $\frac{1}{x}$.

Hence, the reciprocal of a a fraction is the fraction inverted.

Thus, the reciprocal of $\frac{2}{3}$ is $1 \div \frac{2}{3}$; that is, $1 \times \frac{3}{2}$, or $\frac{3}{2}$.

Similarly, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; of $\frac{a-b}{x+y}$ is $\frac{x+y}{a-b}$.

EXERCISE 56

Simplify:

1. $\frac{5x^2y}{14a^3c} \times \frac{28a^2b^3}{15xy^2}$

2. $\frac{21xy^2}{13z^3} \div \frac{28x^3}{39z^4}$

3. $\frac{9a^2b}{8c^2x} \times \frac{28ax^2}{15b^2c} \div \frac{21a^3x}{10bc^3}$

4. $\frac{50x^2z}{49y^n} \times 28xy^{3n} \times \frac{-y^n}{40x^3}$

5. $\frac{15x}{2x(2x-1)} \times \frac{2x(x+1)}{5x^2}$

6. $\frac{a^2b^2+3ab}{4a^2-1} \div \frac{ab+3}{2a+1}$

7. $\frac{x^2-9}{x^2+x} \div \frac{x-3}{x^2-1}$

8. $\frac{(a-1)^3}{a(x+1)^2} \times \frac{x+1}{(a-1)^2}$

9. $\frac{4x^2-9}{9x^2-1} \times \frac{6x+2}{12x-18}$

10. $\frac{2x^2-x-1}{2x^2+x-1} \times \frac{4x^2-1}{x^2-1}$

11. $\frac{a^3y-ax^2y}{a^3x^2+a^2x^2y} \div \frac{a^2y-2axy+x^2y}{a^2+ay}$

12. $\frac{a^3-1}{(a+1)^2} \div \frac{a^3-a}{(a+1)^3}$

13. $\frac{3x^2+x-2}{4x^2-4x-3} \times \frac{4x^2-1}{9x^2-4}$

14. $\left(x+\frac{1}{x-1}\right) \times \frac{2x-2}{x^3+1}$

15. $\left(\frac{x}{y}+1\right) \div \left(\frac{x^2}{y}+\frac{y^2}{x}\right)$

16. $\frac{x^2-y^2}{5xy} \times \frac{x-y}{x+y} \times \frac{10x^2y^2}{(x-y)^2} \times \frac{1}{2xy}$

17. $\frac{3(a-b)^2}{4(a+b)^2} \times \frac{7(a^2-b^2)}{9(a-b)^3} \div \frac{14ab}{8(a+b)}$

18. $\frac{x^2+2x-3}{x^2+x-12} \times \frac{x^2+2x-15}{x^2+2x-3} \div \frac{x^3+5x^2}{x^3+4x^2}$

19. $\frac{6x^2y-4xy^2}{45x^2-20y^2} \times \frac{30x+20y}{4x^2y^2} \times \frac{xy}{x+y}$

20. $\frac{6x^2-5x-4}{2x^2+7x-4} \times \frac{6x^2+x-2}{4x^2-4x-3} \times \frac{2x^2+5x-12}{9x^2-6x-8}$

21. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right) \div \frac{x^6-1}{x^2(x^2-1)}$

22. $\left(\frac{a}{b}+1\right)\left(1-\frac{b}{a}\right)\times\left(1-\frac{a^2+b^2}{a^2-b^2}\right)$

23. $\frac{x^3+y^3}{x^3+x^2y+xy^2}\times\left(1+\frac{y}{x-y}\right)\div\frac{x^2-xy+y^2}{x^3-y^3}$

24. $\frac{x^2-(a-1)^2}{a^2-(x+1)^2}\times\frac{(a+x)^2-1}{1-(a-x)^2}\div\frac{a+x-1}{a-x-1}$

25. $\frac{2-b-a}{b-2-a}\times\frac{a^2-b^2-4b-4}{a^2+b^2+2ab-4}\times\frac{b^2-a^2-4b+4}{b^2-a^2+4a-4}$

26. $\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\div\frac{(a+b+c)^2}{a^2b^2c^2}$

27. $\left(\frac{1}{x}-x^2\right)\left(x+\frac{x^2}{1-x}\right)\div\left[(x+1)^2-x\right]$

28. $\left(\frac{1}{a^2}+\frac{1}{x^2}+\frac{2}{ax}-1\right)\div\left[\frac{x+a(1-x)}{ax}\times\left(1+\frac{a}{x}+a\right)\right]$

29. $\left[\frac{m+2n}{m-2n}+\frac{m-2n}{m+2n}\right]\div\left[\frac{m+2n}{m-2n}-\frac{m-2n}{m+2n}\right]$

30. Write the reciprocal of each of the following: 3, a, $2x$, $\frac{4}{5}$, $\frac{1}{5}$, $\frac{a}{2x}$, $\frac{1}{2x}$, $\frac{a+2x}{a-2b}$, $\frac{1}{a^2-2b}$.

31. Make up and work two examples involving both multiplication and division of fractions.

32. How many of the examples in Exercise 15 (p. 60) can you now work at sight?

IV. Reduction of Complex Fractions

123. A Complex Fraction is one having a fraction in its numerator, or in its denominator, or in both.

In simplifying any complex fraction, it is important to write the entire fraction at each step of the process.

Ex. 1 $\dfrac{x}{1-\dfrac{x}{y}} = \dfrac{x}{\dfrac{y-x}{y}} = x \times \dfrac{y}{y-x} = \dfrac{xy}{y-x}$ *Ans.*

When the numerator and denominator of a complex fraction each contain fractions, the expression is often simplified most readily if we

Multiply both numerator and denominator by the lowest common denominator of the fractions contained in them.

Ex. 2. Simplify $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}}$.

Multiplying both numerator and denominator by xyz, obtain

$$\frac{yz+xz+xy}{x^2z+xy^2+yz^2} \text{ Ans.}$$

124. A Continued Fraction is a fraction whose denominator is a mixed expression, having another mixed expression in *its* denominator, and so on until the fraction ends.

Ex. Simplify $\dfrac{1}{x+\dfrac{1}{1-\dfrac{3}{x-2}}}$.

$$\frac{1}{x+\dfrac{1}{\left[1-\dfrac{3}{x-2}\right]}} = \left.\frac{1}{x+\dfrac{1}{\left[\dfrac{x-5}{x-2}\right]}}\right\} = \frac{1}{x+\dfrac{x-2}{x-5}}$$

$$= \frac{1}{\dfrac{x^2-4x-2}{x-5}} = \frac{x-5}{x^2-4x-2} \text{ Ans.}$$

Hence, in general, to simplify a continued fraction,

Reduce the last mixed expression in the fraction to an improper fraction (see the brackets in the examples);

Then invert the last fraction and multiply it into the numerator under which it is placed (see the brace);

Thus alternately reduce a mixed number and invert a divisor fraction until the simplification is completed.

EXERCISE 57

Simplify:

1. $$\frac{\frac{4}{x}-x}{1+\frac{x}{2}}$$

3. $$\frac{x-\frac{1}{x}}{1-\frac{1}{x}}$$

5. $$\frac{1-\frac{1}{a+1}}{1+\frac{1}{a-1}}$$

2. $$\frac{2-\frac{1}{x}}{4-\frac{1}{x^2}}$$

4. $$\frac{\frac{ab}{c}-2d}{a-\frac{2cd}{b}}$$

6. $$\frac{x-\frac{2}{3}}{6}$$

7. $$\frac{2x-\frac{1}{4x^2}}{1-\frac{1}{2x}}$$

11. $$\frac{\frac{2x}{y}+1-\frac{y}{x}}{\frac{2x}{y}+\frac{y}{x}-3}$$

8. $$\frac{\frac{1}{x}-\frac{3}{x^3}-\frac{2}{x^2}}{\frac{9}{x^2}-1}$$

12. $$\frac{\frac{x}{1+x}+\frac{1-x}{x}}{\frac{x}{1+x}-\frac{1-x}{x}}$$

9. $$\frac{\left(\frac{1}{x}+\frac{1}{y}\right)^2}{\left(1+\frac{x}{y}\right)^2}$$

13. $$\frac{a+\frac{1}{a+1}-1}{a+\frac{1}{a-1}+1}$$

10. $$\frac{\frac{a}{x^2}+\frac{x}{a^2}}{\frac{1}{a^2}-\frac{1}{ax}+\frac{1}{x^2}}$$

14. $$\frac{\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)^2}{\frac{c^2(a+b)^2-a^2b^2}{a^2b^2c^2}}$$

15. $$\dfrac{\dfrac{x}{a}+\dfrac{a}{x}-2-\dfrac{1}{ax}}{\dfrac{x}{a}-\dfrac{a}{x}-\dfrac{2}{a}+\dfrac{1}{ax}}$$

16. $$\frac{2(\frac{1}{2}x-\frac{1}{4})}{2x-1}-\frac{1}{2}$$

17. $$\dfrac{\dfrac{1-(ab-cd)^2}{(ab-1)^2-c^2d^2}}{\dfrac{(cd+1)^2-a^2b^2}{(ab+cd)^2-1}}$$

18. $$3-\frac{x-1}{2x-\frac{5x}{3}}$$

19. $$1-\cfrac{1}{1+a+\cfrac{2a^2}{1-a}}$$

20. $$3-\cfrac{1}{a-\cfrac{a^2}{1+a}}$$

21. $$2a-1-\cfrac{a-1}{2-\cfrac{a}{a-\cfrac{a}{1+a}}}$$

22. $$3-\cfrac{1}{1-\cfrac{2x}{3x-\cfrac{3x}{x+1}}}$$

23. $$1-\dfrac{a-b}{\dfrac{2}{a+b}}$$

24. $$1-\dfrac{\dfrac{a-b}{2}}{a+b}$$

25. $$\dfrac{1-\dfrac{a^3}{8}}{1+\dfrac{a}{2}+\dfrac{a^2}{4}}\div(a-2)$$

26. $$\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\times\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

27. $$\cfrac{1}{x+\cfrac{1}{x+2}}\times\cfrac{1}{x+\cfrac{1}{x-2}}\div\dfrac{x-\dfrac{4}{x}}{x^2+\dfrac{1}{x^2}-2}$$

28. $$\cfrac{1+x^3}{1-\cfrac{x}{1+\cfrac{x}{1-x}}}-\cfrac{1-x^3}{1+\cfrac{x}{1-\cfrac{x}{1+x}}}$$

29. $\dfrac{x^3y - y^4}{xy^2 + x^2y} \div \left\{\dfrac{x^4 + x^3y + x^2y^2}{(x^2 - y^2)^3} \div \dfrac{1}{\left(1 + \dfrac{y}{x}\right)^2}\right\}.$

30. $\dfrac{\dfrac{1}{a^2} - \left(\dfrac{1}{b} + \dfrac{1}{c}\right)^2}{\dfrac{1}{b^2} - \left(\dfrac{1}{a} + \dfrac{1}{c}\right)^2} \times \dfrac{\dfrac{1}{b^2} - \left(\dfrac{1}{a} - \dfrac{1}{c}\right)^2}{\dfrac{1}{c^2} - \left(\dfrac{1}{a} - \dfrac{1}{b}\right)^2} \times \dfrac{\left(\dfrac{1}{a} + \dfrac{1}{c} - \dfrac{1}{b}\right)^2}{\left(\dfrac{1}{a} - \dfrac{1}{c}\right)^2 - \dfrac{1}{b^2}}.$

31. $\dfrac{1 - 2\dfrac{1 - 2x}{1 + 2x}}{1 + 2\dfrac{1 + 2x}{1 - 2x}} + \dfrac{4(\frac{1}{2} - \frac{1}{2}x + x^2) - \frac{2}{3}}{\frac{8}{3}(\frac{1}{2} + x + \frac{1}{2}x^2) - \frac{1}{3}}.$

Find the value

32. Of $\dfrac{22}{1 + 2v}$ when $v = \dfrac{2}{7}.$

33. Of $\dfrac{-22}{1 - v - v^2}$ when $v = -\dfrac{1}{3}.$

34. Of F when $F = \dfrac{mv^2}{r}$, $m = 10.2$, $v = 5$, and $r = \dfrac{5}{2}.$

35. Make up and simplify a continued fraction.

36. How many examples in Exercise 19 (p. 78) can you now work at sight?

EXERCISE 58

ORAL REVIEW

1. Give the value of each of the following:

(1) $\dfrac{1}{2a} - \dfrac{1}{4a}.$ (2) $\dfrac{1}{3a} - \dfrac{1}{6a}.$ (3) $\dfrac{a + b}{2} + \dfrac{a - b}{2}.$

(4) $\dfrac{a + b}{2} - \dfrac{a - b}{2}.$ (5) $\dfrac{1}{a} + \dfrac{1}{b}.$ (6) $\dfrac{1}{a} - \dfrac{1}{b}.$

2. Expand (1) $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)^2$ (2) $\left(\dfrac{2}{x} - \dfrac{3}{y}\right)^2$ (3) $\left(\dfrac{1}{a} - \dfrac{2}{b}\right)^2$

3. On the foot rule show the meaning of $\frac{1}{4}$ in. ÷ 2. Of $\frac{3}{4}$ in. ÷ 2.

4. Divide each of the following fractions by 2: $\frac{4}{5}$, $\frac{3}{5}$, $\frac{7}{8}$, $\frac{9}{5}$, $\frac{8}{7}$, $\frac{2a}{b}$, $\frac{a}{2b}$, $\frac{a}{b}$, $\frac{a+b}{2}$, $\frac{6}{a}$, $\frac{7}{a}$, $\frac{3x}{2b}$, $\frac{5x}{4b}$, $\frac{4b^2}{a}$, $\frac{3b^2}{a}$.

5. Divide $\frac{3x}{b}$ by 2. By $2a$. $3b$. $4b$.

6. Divide 1 by each of the fractions $\frac{1}{2}$, $\frac{3}{2}$, $\frac{a}{2}$, $\frac{a}{b}$, $\frac{3a}{2b}$.

7. Give the reciprocal of 3, $\frac{1}{3}$, 4, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{a}{b}$, $\frac{a}{b}$.

8. Give the value of $\frac{1}{x}$ when $x = \frac{1}{2}$. When $x = \frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{2}{3}$, $\frac{3}{8}$, $\frac{2}{5}$, $-\frac{2}{3}$, $-\frac{1}{a}$, $\frac{b}{c}$.

9. State the value of $\frac{3}{x}$ when $x = \frac{1}{2}$. When $x = \frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $-\frac{2}{3}$, $\frac{3}{2}$, $\frac{6}{5}$.

10. Give the value of $\frac{15}{2+x}$ when $x = -\frac{1}{5}$.

11. What is $\frac{1}{2}$ of $\frac{4}{5}$? Of $\frac{3}{5}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{a-1}{2}$, $\frac{a}{b}$, $\frac{4a}{b}$, $\frac{1}{2a}$, $\frac{2a}{3}$, $\frac{1}{4ab}$?

12. If 4 is subtracted from both numerator and denominator of $\frac{5}{10}$, is the value of the fraction changed? By how much?

13. If $a = -\frac{3}{2}$ and $b = -\frac{1}{2}$, state the value of $\frac{-a}{b}$. Of $-\frac{3a}{4b}$.

14. What is the value of $1 \div 2/3$? Of $1 \div a/b$? Of $2 \div x/2y$?

15. Simplify those of the following fractions which can be reduced to lower terms:

$$\frac{4x}{4a+b},\ \frac{4x}{4a+4b},\ \frac{4a}{4b},\ \frac{4}{a+4},\ \frac{a-b}{b-a},\ \frac{a+b}{a^2+b^2},\ \frac{a^2+b^2}{a^2+x^2},\ \frac{a^2b}{a^2x}.$$

Give the value of

16. $-\frac{-3-1}{8}$, $\frac{-2}{-4-6}$, $\frac{8}{-2/3}$, $\frac{8}{a/b}$, $\frac{x^2-1}{1-x}$.

17. Of $\frac{1}{4}a^3 \div \frac{1}{2}a^2$ $\quad \frac{1}{4}a^3 \div \frac{1}{3}a^2$ $\quad \frac{2}{3}x^4 \div \frac{1}{3}x^2$ $\quad \frac{3}{5}x^3 \div \frac{2}{5}x$ $\quad 1 \div \frac{1}{3}x^2$
$1 \div \frac{2}{3}x^2$ $\quad 2a^2 \div \frac{3}{4}a$ $\quad 2 \div 5/6x$

18. Make up an illustration to show the value of 5×0 (for instance, in connection with a pupil's mark for 5 examples which he failed to work correctly).

19. Give in the briefest form the product of $(a - b)(b - a)$. Of $(x - 2y)(2y - x)$. Of $(a - b)(a + b)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)$.

20. Does $\frac{1}{(y - x)^3} = -\frac{1}{(x - y)^3}$?

EXERCISE 59

WRITTEN REVIEW

1. Indicate by a parenthesis that $2a - 3b + c$ is to be subtracted from $5a + 2b - 3c$. Then remove the parenthesis and simplify.

2. Subtract the sum of $3x + 2y - z$ and $a - x - 3y$ from -5. Also from 0.

3. Write by inspection the value of $[(3a - b) - c + 2d]^2$.

4. Factor $(a + b - c)^3 - (x + y - x)^3 \cdot \frac{1}{x^3} - \frac{1}{y^3} \cdot 1 - \frac{x^3}{8}$

5. Change $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2}$ so that it shall be a perfect square.

6. What is the difference between an exponent and a power? Give an illustration.

7. Subtract $(5a + 1)(2a - 3)$ from $(a + 2)(a + 1) + (a + 2)^2$.

8. Find the value of $3(x - 1)^2 - 3(x + 1)(x + 2) - x(x - 2)(y - 2x)$ when $x = -2$ and $y = -5$.

9. Find the value of $\frac{2A^2 - B^2}{3B}$ when $A = 5a$ and $B = 2a$.

10. By factoring find the roots of $x^2 - 5x + 6 = 0$. Prove your answer.

11. Show that $\frac{a - b}{c - d}$ is equal to $\frac{b - a}{d - c}$.

12. If $a = 12\frac{1}{2}$, $b = 37\frac{1}{2}$, $c = 33\frac{1}{3}$, and $d = 10$, find in the shortest way the numerical value of each of the following: $\frac{4d^2}{a}$, $\frac{4d^2}{c}$, $\frac{4d^2}{b}$.

13. From $7.08a^2$ take $-4\frac{1}{4}a^2$.

14. Reduce $-1 + \frac{(a^2 + b^2)^2}{4a^2b^2}$ to an improper fraction.

15. When we change $x - 3 = 5$ to $x = 5 + 3$, what is the change called? What right have we to make this change? Why do we transpose -3 instead of adding 3 to each member of the given equation?

16. What is the use or advantage in being able to find the H. C. F. of two given expressions? In being able to find their L. C. M.? Illustrate.

17. Show that the sum of two numbers (as of a and b), divided by the sum of their reciprocals, equals the product of the given numbers.

18. If $s = \frac{a}{1 - r}$, find the value of s when $a = 2$ and $r = -\frac{1}{2}$. Also when $a = -\frac{2}{3}$ and $r = -\frac{3}{4}$.

19. If $s = \frac{rl - a}{r - 1}$, find the value of s when $r = \frac{1}{2}$, $l = \frac{31}{2}$, and $a = -\frac{1}{2}$.

20. If $a = 3$, which is greater, $\frac{a}{10 - a}$ or $\frac{a + 1}{3a}$?

21. Divide $2a^3 + 10 - 16a - 39a^2 + 15a^4$ by $2 - 5a^2 - 4a$.

22. Give an illustration to show why 3×0 gives zero. Also why $\frac{0}{3}$ gives zero.

23. Show that a common factor of any two algebraic expressions is also a common factor of their sum. Also of their difference. Of the sum or difference of any multiples of the given expressions.

24. Prove that if half of the sum of any two numbers (as of a and b) is added to half their difference, the result will equal the greater of the two numbers. Illustrate by two numerical examples.

25. Prove that if half the difference of any two numbers is subtracted from half their sum, the result will be the smaller of the two numbers.

26. Write an example of a continued fraction and reduce it.

27. Why is it allowable to change both minus signs to plus in $-x = -3$, and not in $-x - 3$?

28. Collect in a short way $\frac{1}{x + 2} + \frac{1}{x - 2} + \frac{1}{x - 3} + \frac{1}{x + 3}$.

Sug. Collect the first two fractions first.

29. Collect in a short way $\frac{3}{x+2} - \frac{2}{x-3} - \frac{1}{x-2} + \frac{4}{x+3}$.

30. Also $\frac{x}{x-y} - \frac{x}{x+y} - \frac{2xy}{x^2+y^2} - \frac{4xy^3}{x^4+y^4}$.

Simplify:

31. $\frac{\frac{3}{2}(\frac{2}{3}x^2+\frac{1}{3}x-2)}{\frac{2}{3}(\frac{3}{2}x^2+\frac{5}{2}x-1)}$.

32. $\frac{2-x}{1-2x} - \frac{x+2}{2x+1} + \frac{6x}{4x^2-1}$.

33. $\frac{\frac{x}{y}\left(\frac{y^3}{x} + xy - 2y^2\right)}{x^2+xy-2y^2}$.

34. $\frac{1}{x-1} + \frac{2}{x-2} - \frac{3}{x-3} + \frac{4x-3}{(x^2-x)(x-2)}$.

35. $\frac{1}{a-\frac{1}{a}} + \frac{1}{a+\frac{1}{a}} - \frac{2}{a-\frac{1}{a^3}} + \frac{b}{ab+\frac{b}{a}}$.

36. $\frac{3}{x^2-3x+2} + \frac{2}{(x-1)(3-x)} + \frac{1}{(2-x)(x-3)}$.

37. $\left(\frac{a}{x}+\frac{x}{a}-2\right)\left(\frac{a}{x}+\frac{x}{a}+2\right) \div \left(\frac{a}{x}-\frac{x}{a}\right)^2$.

38. $\left\{\frac{1+x}{1+x^2} - \frac{1+x^2}{1+x^3}\right\} \div \left\{\frac{1+x^2}{1+x^3} - \frac{1+x^3}{1+x^4}\right\}$.

39. $\left\{\frac{a+\frac{1}{x}}{a-\frac{x^2}{a}}\right\}^2 \times \left\{\frac{1+\frac{a}{x}}{1+\frac{1}{ax}}\right\}^2 \times \left(\frac{1}{a}-\frac{x}{a^2}\right)^2$.

40. $\frac{x-4-\frac{4}{x-4}}{x-4-\frac{1}{x-4}} \times \frac{x-\frac{1}{x-2}-2}{x-\frac{4}{x-5}-2}$.

41. $\left\{1 - \frac{\frac{9}{x^2}+\frac{x^2}{9}}{\frac{9}{x^2}-\frac{x^2}{9}}\right\} \times \left(\frac{3}{x}-\frac{x}{3}\right)\left(\frac{9}{x^2}+1\right)$.

42. $$\frac{1+8x^3}{1-\cfrac{2x}{1+\cfrac{2x}{1-2x}}}-\frac{1-27x^3}{1+\cfrac{3x}{1-\cfrac{3x}{1+3x}}}$$

43. $$\frac{x^4-5x^2+4}{x^2+1}\times\frac{\frac{1}{x}-\frac{1}{x-2}}{x-\frac{1}{x}}\div\frac{2+\frac{4}{x}}{1+\frac{1}{x^2}}.$$

44. Given $a+b+c=2s$, show that $a+b-c=2(s-c)$ and that $a-b+c=2(s-b)$.

45. Also show that

$$1-\frac{a^2+c^2-b^2}{2ac}=\frac{2(s-a)(s-c)}{ac}.$$

46. Also show that $1+\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{2s(s-c)}{ab}.$

47. Show that

$$\frac{(2a-3x)^9\,[8x^5(a+2x)^3+5x^4(a+2x)^4]+27x^5(a+2x)^4(2a-3x)^8}{(2a-3x)^{18}}$$

reduces to

$$\frac{2ax^4(24x+5a)(a+2x)^3}{(2a-3x)^{10}}.$$

48. The distance from New York to San Francisco by way of Cape Horn is 13,800 mi. This is 1920 mi. less than three times the distance from New York to San Francisco by way of Panama. Find the latter distance.

49. Make up and work an example similar to Ex. 48, using the fact that the distance from London to Bombay by way of the Cape of Good Hope is 11,220 mi., but by way of the Suez Canal is 6332 mi.

CHAPTER XI

FRACTIONAL AND LITERAL EQUATIONS

125. A fractional equation is an equation that contains an unknown number in a denominator.

Ex. $\frac{8}{x} + 5 = 3x.$

Equations containing binomial numerators and numerical denominators are frequently termed fractional equations, since they are solved in the same manner as fractional equations proper. See Ex. 1. of Art. 126.

An **integral equation** is an equation which does not contain an unknown number in a denominator.

126. The Method of Solving a Fractional Equation. If an equation contains fractions, it is necessary first to multiply the members of the equation by such a number as will remove the fractions.

Ex. 1. Solve $\frac{x+1}{2} - \frac{2x-5}{5} = \frac{11x+5}{10} - \frac{x-13}{3}.$

The L. C. D. of the denominators is 30.

Multiplying both members of the equation by 30 (see Art. 70, 3), we have

$$\begin{aligned} 15(x+1) - 6(2x-5) &= 3(11x+5) - 10(x-13) \\ \text{Hence,}\quad 15x + 15 - 12x + 30 &= 33x + 15 - 10x + 130 \\ 15x - 12x - 33x + 10x &= -15 - 30 + 15 + 130 \\ -20x &= 100 \\ x &= -5 \ \textit{Root} \end{aligned}$$

CHECK. $\frac{x+1}{2} - \frac{2x-5}{5} = \frac{-5+1}{2} - \frac{-10-5}{5} = -2 + 3 = 1$

$$\frac{11x+5}{10} - \frac{x-13}{3} = \frac{-55+5}{10} - \frac{-5-13}{3} = -5 + 6 = 1$$

Ex. 2. Solve $\frac{4}{1+x}+\frac{x+1}{1-x}-\frac{x^2-3}{1-x^2}=0.$

Multiplying by the L. C. D., $1-x^2$,

$$\begin{aligned} 4(1-x)+(x+1)^2-x^2+3&=0\\ 4-4x+x^2+2x+1-x^2+3&=0\\ -2x&=-8\\ x&=4\ \textit{Root}\end{aligned}$$

Let the pupil check the work.

Hence, in general,

Reduce each fraction in the equation to its lowest terms;

Clear the equation of fractions by multiplying each member by the L. C. D. of all the fractions;

Complete the solution by the methods of Chapter VI.

EXERCISE 60

Solve and check each result:

1. $\frac{x}{3}-\frac{3x}{5}+\frac{7x}{5}=\frac{34}{15}.$

2. $\frac{2x-3}{4}+\frac{x+1}{6}=\frac{5x+2}{12}.$

3. $\frac{1}{2x}-\frac{3}{x}+\frac{5}{3x}=\frac{3}{4x}-\frac{19}{24}.$

4. $\frac{2x}{3}-\frac{2x+1}{5}=\frac{1}{3}.$

5. $\frac{3x+5}{4}=1-\frac{x+4}{6}.$

6. $7=\frac{3}{5}(x-2).$

7. $2x-8-\frac{1}{7}(24-2x)=0.$

8. $\frac{3}{4}(x-1)=\frac{1}{3}(x-2).$

9. $3(\frac{2}{3}x-\frac{1}{2})(\frac{1}{2}x+\frac{2}{3})=x^2.$

10. $\dfrac{3-2x}{8}-\dfrac{x-3}{6}-1=\dfrac{x+4}{3}+\dfrac{1}{24}.$

11. $\dfrac{3x-1}{7}-\dfrac{x+1}{6}-\dfrac{4x+1}{21}=\dfrac{3(x-1)}{4}-3.$

12. $\dfrac{2x+5}{5}-\dfrac{x+1\frac{1}{2}}{10}+x=\dfrac{5x-10\frac{1}{2}}{20}-\dfrac{1}{5}.$

13. $\dfrac{3-x}{10}-\tfrac{2}{3}(5+x)+\dfrac{x-3}{5}-\tfrac{1}{2}(2x+5)=\dfrac{7x-4}{6}.$

14. $2(x+\tfrac{1}{3})+x\left(1-\dfrac{1}{2x}\right)=\dfrac{6x+5}{12}+\dfrac{x+5}{4}.$

15. $\dfrac{x+5}{7}-\dfrac{x+7}{5}+\dfrac{x+1}{2}-\dfrac{2x-5}{10}=\dfrac{x+22}{70}.$

16. $\tfrac{2}{3}(5x+2)-\tfrac{3}{4}(7x-2)+\tfrac{1}{2}(3x-2)=x-\dfrac{x}{2}.$

17. $.5x-.4x=.3.$

18. $1.5x-5=x.$

19. $.6x-1.5=.2-.15x.$

20. $\dfrac{1.5x-1.6}{1.2}=\dfrac{3.5x-2.4}{.8}.$

21. $\dfrac{3.2x-3.4}{4.5}=\dfrac{.6x+4}{2.5}.$

22. $\dfrac{x-1}{x+1}=\dfrac{3}{5}.$

23. $\dfrac{5}{2x-1}=\dfrac{8}{3x+1}.$

24. $\dfrac{6x-5}{3x-3}=\dfrac{8x-7}{4x+4}.$

25. $\dfrac{x}{3}-\dfrac{x^2-5x}{3x-7}=\dfrac{2}{3}.$

26. $\dfrac{5}{1-x}+\dfrac{6}{1+x}=\dfrac{7}{1-x^2}.$

27. $\dfrac{3}{3-x}+\dfrac{4}{3+x}=\dfrac{8x+3}{9-x^2}.$

28. $\dfrac{2x+1}{2x-1}-\dfrac{10}{4x^2-1}=\dfrac{2x-1}{2x+1}.$

29. $\dfrac{1}{x+1}+\dfrac{1}{x-1}=\dfrac{2}{x+2}.$

30. $\dfrac{x-1}{x^3-8} = \dfrac{3}{x-2} + \dfrac{1-3x}{x^2+2x+4}.$

31. $\dfrac{x^2-x+1}{x-1} = 2x - \dfrac{x^2+x+1}{x+1}.$

32. $\dfrac{x-3}{2x+6} - \dfrac{\frac{1}{6}x^2+1}{x^2-9} = \dfrac{x+3}{3x-9}.$

33. $\dfrac{3}{x-1} + \dfrac{4}{x+1} - \dfrac{5}{2x-2} = \dfrac{11}{3x+3} + \dfrac{4}{1-x^2}.$

34. $\dfrac{x+1}{2x-3} - \dfrac{x^2+7}{4x^2-9} = \dfrac{2}{2x+3} - \dfrac{x-1}{6-4x}.$

35. $\dfrac{3x^2-5}{3x-6} - \dfrac{7}{6x+12} - 2 - x = \dfrac{7}{2x^2-8}.$

36. $\dfrac{6x+6}{2x^2+5x+3} - \dfrac{2x+1}{2x^2-x-1} = \dfrac{2x}{x^2+2x}.$

Reduce each fraction in the following to its lowest terms and then solve:

37. $\dfrac{x}{2} = \dfrac{x^2-1}{x+1}.$

38. $\dfrac{4x+4}{x^2+x} - \dfrac{3}{4} = 0.$

39. $\dfrac{x^2-x-2}{x-2} = 4 - 6(x-3).$

40. $\dfrac{x^3-8}{x^2+2x+4} = 8-3(x+4).$

Find the value of the letter in each of the following:

41. $\dfrac{1}{v+2} + \dfrac{7}{3(v+2)} = \dfrac{2}{3}.$

42. $\dfrac{1}{3(p-7)} + \dfrac{1}{6} = \dfrac{1}{2p-14}.$

43. $\dfrac{15}{3-2s} - \dfrac{2}{6-4s} - 14 = 0.$

44. $\dfrac{4}{2t+2} + \dfrac{31}{3t+3} = \dfrac{1}{6}.$

45. $\dfrac{r+6}{11} - \dfrac{2r-18}{3} + \dfrac{2r+3}{4} = 5\frac{1}{3} + \dfrac{3r+4}{12}.$

46. If $A = lw$, $A = 600$ and $w = 20$, find the value of l.

Do you know the meaning of this process in arithmetic in connection with the rectangle?

47. In like manner, find l when $A = 80$ and $w = 11\frac{2}{3}$.

48. If $V = lwh$, $V = 720$, $l = 10$, and $w = 6$, find h.

Do you know the meaning of this process in arithmetic in connection with the study of volumes?

49. In like manner if $V = .36$, $w = .8$, and $h = .9$, find l.

50. If $p = br$, $p = 9$ and $b = 45$, find r.

Do you know the meaning of this process in arithmetic in connection with the subject of percentage?

51. If $p = br$, $p = 760$ and $r = .05$, find b.

52. If $i = prt$, $i = \$66$, $p = \$440$ and $t = 3$, find r.

Do you know the meaning of this process in arithmetic in connection with the subject of interest?

53. If $i = prt$, $i = \$66$, $p = \$360$, and $t = 3\frac{2}{3}$, find r.

54. If $i = prt$, $i = \$15.75$, $p = \$75$, $r = .06$, find t.

55. If $C = \frac{5}{9}(F - 32)$ find F when $C = 50$. When $C = 100$.

Do you know the meaning of this process?

56. If $LW = lw$ and $L = 8$, $W = 100$, and $w = 40$, find l.

Can you find out the meaning of this formula and process?

57. If $R = \dfrac{gs}{g + s}$, find s when $R = 10$ and $g = 32$.

58. If $V = \left(1 + \dfrac{t}{273}\right)v$, find v when $V = 20$ and $t = 13$.

59. If $K = \frac{1}{2}h(b + b')$, $K = 280$, $h = 12$, and $b = 10$, find b'.

60. If $V = \pi R^2 H$, $V = 1540$, $\pi = \frac{22}{7}$, and $R = 7$, find H.

61. If $T = \pi R(R + L)$, $T = 1144$, $\pi = \frac{22}{7}$, $R = 14$, find L.

62. Make up and work an equation containing fractions with the denominators 4, 6, and 12. Can you form the equation so that the root shall be 1? 2? 4?

63. Make up and work an equation containing fractions with the denominators $x + 2$, $x - 2$, and $x^2 - 4$.

64. Work again Exercise 24 (p. 99).

127. Special Methods. The work of solving an equation may often be lessened by using some special method or device adapted to the peculiarities of the given equation.

First Special Method. If in a given equation *the denominators of some fractions are monomials, and of others are polynomials,* it is best to make two steps of the process of clearing the equation of fractions: (1) remove the monomial denominators and simplify as far as possible; (2) remove the remaining polynomial denominators.

Ex. 1. Solve $\frac{2x + 8\frac{1}{2}}{9} - \frac{13x - 2}{17x - 32} + \frac{x}{3} = \frac{7x}{12} - \frac{x + 16}{36}$.

Multiplying by 36, the L. C. D. of the monomial denominators,

$$8x + 34 - \frac{36(13x - 2)}{17x - 32} + 12x = 21x - x - 16$$

Transposing all terms except the fraction to right-hand side,

$$-\frac{36(13x - 2)}{17x - 32} = -50$$

Dividing by -2, $\frac{18(13x - 2)}{17x - 32} = 25$

$$234x - 36 = 425x - 800$$

$$191x = 764$$

$$x = 4 \quad \textit{Root}$$

Let the pupil check the work.

Second Special Method. Before clearing an equation of fractions, it is often best to combine some of the fractions into a single fraction.

Ex. 2. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$.

In this equation it is best to combine the fractions in the left-hand member into a single fraction, and those in the right-hand member also into a single fraction, before clearing of fractions. We obtain

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-4)(x-5)}$$

Clearing and solving, $x=\frac{7}{2}$ *Root*

Let the pupil check the work.

EXERCISE 61

Solve the following and check each solution:

1. $\frac{3x-1}{6}+\frac{4x}{3x+2}=\frac{x+5}{2}$.

2. $\frac{3-2x}{4}+\frac{x}{6}-\frac{1-6x}{15-7x}=\frac{2-3x}{9}$.

3. $2\frac{1}{2}-\frac{3}{2x+4}=\frac{2x-1}{4}-\frac{x}{2}$.

4. $\frac{5x+13}{12}=\frac{2x+5}{6}+\frac{23}{4x-36}-\frac{5-\frac{1}{4}x}{3}$.

5. $\frac{5}{7-x}-\frac{2\frac{1}{4}x-3}{4}-\frac{x+11}{8}+\frac{11x+5}{16}=0$.

6. $\frac{3x-1}{30}+\frac{4x-7}{15}=\frac{x}{4}-\frac{2x-3}{12x-11}+\frac{7x-15}{60}$.

7. $\frac{6x-7}{11x+5}-\frac{x+1}{15}+\frac{2x-1}{30}=\frac{199}{10}$.

8. $3+\frac{x+4}{7x+11}=\frac{4-3x}{8}+\frac{4x+9}{12}-\frac{4-x}{24}+\frac{5}{4}$.

9. $\frac{3x-2\frac{1}{2}}{9}-\frac{7}{12}+\frac{x-\frac{1}{2}}{\frac{4}{3}x+11}=\frac{2x-1\frac{1}{2}}{3}-\frac{2x+3\frac{1}{2}}{6}$.

10. $\frac{2}{3}\left[\frac{2x}{9}-\frac{1}{2}\left\{\frac{3x-1}{6}+\frac{2x-5}{7x+8}+x\right\}\right]+\frac{19x+3}{54}=0.$

11. $\frac{1.2x-1.5}{1.5}+\frac{.4x+1}{.2x-.2}=\frac{.4x+1}{.5}.$

12. $\frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-4}-\frac{1}{x-5}.$

13. $\frac{x-1}{x-2}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-7}{x-8}.$

14. $\frac{x-7}{x-8}-\frac{x-8}{x-9}=\frac{x-4}{x-5}-\frac{x-5}{x-6}.$

15. $\frac{3}{3x-2}-\frac{2}{2x-3}=\frac{2}{2x+3}-\frac{3}{3x+2}.$

16.* $\frac{2x+1}{x+1}+\frac{2x+9}{x+5}=\frac{2x+3}{x+2}+\frac{2x+7}{x+4}.$

17. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}-\frac{8x-30}{2x-7}-\frac{5x-4}{x-1}=0.$

18. Work Ex. 2 by clearing all denominators at once. Then work the same example by the method of Art. 126. About what fraction of the work is saved by the second process?

19. Treat Ex. 13 in the same way as you treated Ex. 2.

20. On the average, the distance one must go below the surface of the earth to get an increase of 1° in temperature, is 62 ft. This is 1 ft. more than one third the distance one must go above the earth's surface to get a decrease of 1° in the temperature. Find the latter distance.

21. Who, so far as we know, first invented transposition in solving equations, and when? Who first brought the use of transposition into prominence?

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* Transpose the second and third fractions.

22. From what language does the word *algebra* come? What does the word *algebra* mean?

23. Work the odd-numbered examples on p. 101. How many examples on that page can you work at sight?

128. Two Equivalent Equations are equations which have identical roots; that is, each equation has all the roots of the other equation and no other roots.

Thus, $x^3 - 4x = 0$, and $x(x + 2)(x - 2) = 0$ are equivalent, since each is satisfied by the values $x = 0, 2, -2$, and by no other values of x.

If we multiply the two members of an equation by the same expression, the resulting members are equal, but the resulting equation may not be equivalent to the original equation.

Thus, if we take the equation $x = 3$ and multiply each member by $x - 2$, we obtain $x(x - 2) = 3(x - 2)$ or

$$(x - 3)(x - 2) = 0,$$

which is not equivalent to the original equation, since it has the root $x = 2$, which the original equation does not have (Art. 103).

In general, *if the two members of an integral equation are multiplied by $x - a$, the root a is introduced* and the resulting equation is not equivalent to the original equation.

129. An Extraneous Root is a root introduced into an equation (usually unintentionally) in the process of solving the equation.

The simplest way in which an extraneous root may be introduced is by multiplying both members of an integral equation by an expression containing the unknown number. See the example of Art. 128.

A more common way in which extraneous roots are introduced during a solution — and one more difficult to detect—

is by a failure to reduce to its lowest terms a fraction contained in the original equation.

Thus, in solving $\frac{2x-4}{(x-1)(x-2)}=1$, the first step should be to reduce the fraction $\frac{2x-4}{(x-1)(x-2)}$ to its lowest terms. If this is done, we obtain the equation $\frac{2}{x-1}=1$, whence $x=3$.

If, however, we should fail to reduce the fraction to its lowest terms and should multiply both members by $(x-1)(x-2)$, we obtain $2x-4=x^2-3x+2$, whence $x^2-5x+6=0$,

$$\text{or } (x-3)(x-2)=0, \text{ and } x=2, 3.$$

On testing both of these results, we find that the extraneous root 2 has been introduced.

Often the fraction which can be reduced to simpler terms occurs in a disguised and scattered form. In this case it is best to solve the equation without attempting to collect the parts of the fraction. An extraneous root may then be detected by checking the results obtained.

Thus, the fraction in the above equation might be changed in the following way so as to make it difficult to detect its presence in the equation:

We have $$\frac{2x-4}{(x-1)(x-2)}=1,$$

whence $$\frac{2x-2}{(x-1)(x-2)}-\frac{2}{(x-1)(x-2)}=1,$$

whence $$\frac{2}{x-2}-\frac{2}{(x-1)(x-2)}=1.$$

There is nothing in the appearance of this last equation to indicate that it implicitly contains a fraction which should be simplified before proceeding with the solution proper.

Hence it is important constantly to remember that a root of an equation is not such because it is the *result of a series of operations*, as clearing an equation of fractions, transposition, etc., but because it *satisfies the original equation.*

130. Losing Roots in the Process of Solving an Equation.

If both members of the equation $(x - 2)(x - 3) = 0$ are divided by $x - 2$, we obtain $x - 3 = 0$.

The resulting equation is not equivalent to the original equation since it does not contain the root $x = 2$, which the original equation contains.

Hence, in general,

If both members of an equation are divided by an expression containing the unknown quantity, write the divisor expression equal to zero, and obtain the roots of the equation thus formed as part of the answer for the original equation.

EXERCISE 62

1. Multiply each member of the equation $x - 2 = 1$ by $x - 2$. Is the resulting equation equivalent to the original equation? Why?

2. Make up and work an example similar to Ex. 1.

3. Multiply each member of the equation $x = 2$ by $x - 5$. Is the resulting equation equivalent to the original equation? Why?

4. Divide each member of the equation $x^2 - 9 = x - 3$ by $x - 3$. Is the resulting equation equivalent to the original equation? Why?

5. Make up and work an example similar to Ex. 4.

6. Solve the equation $\dfrac{x-3}{x^2-9} = 1$ after first reducing the fraction to its lowest terms. Now solve the equation without reducing the fraction to its lowest terms. Do the two methods of solution give the same result? Which result is correct? Why?

7. Make up and work an example similar to Ex. 6.

Solve each of the following, check each result, and point out each extraneous root, giving the probable reason for the occurence of such a root:

8. $\frac{x}{2}+\frac{1}{x+1}=\frac{x^2}{x+1}.$

9. $\frac{3}{x+1}=1-\frac{9}{(x+1)(x-2)}.$

10. $\frac{5}{(x+2)(x+3)}+1=\frac{5}{x+2}.$

11. $\frac{x^3-1}{x^2-1}=x-\frac{7}{6}.$

12. $\frac{2x^2}{x^2-1}+\frac{x}{x-1}-\frac{x}{x+1}=3.$

13. Form an equation in which 3 is the extraneous root.

14. How many examples in Exercise 31 (p. 121) can you now work at sight?

131. A numerical equation is an equation in which the known quantities are expressed by figures. Thus, all the equations on p. 199 are numerical equations.

A **literal equation** is an equation in which some or all of the known quantities are denoted by letters; as by $a, b, c \ldots$, or $m, n, p \ldots$

The methods used in solving literal equations are the same as those used in solving numerical equations.

Ex. Solve $a(x-a)=b(x-b)$.

$$\begin{aligned} ax-a^2&=bx-b^2\\ ax-bx&=a^2-b^2\\ (a-b)x&=a^2-b^2\\ x&=a+b \quad \textit{Root} \end{aligned}$$

CHECK.

$$\begin{aligned} a(x-a)&=a(a+b-a)=ab\\ b(x-b)&=b(a+b-b)=ab \end{aligned}$$

EXERCISE 63

Solve for x and check:

1. $3x + 2a = x + 8a.$
2. $9ax - 3b = 2ax + 4b.$
3. $5ax - c = ax - 5c.$
4. $ax + b = bx + 2b.$
5. $3cx = a - (2b - a + cx).$
6. $5x - 2ax = 3 - b.$
7. $2ax - 3b = cx + 2d.$
8. $(x + a)(x - b) = x^2.$
9. $ab(x + 1) = a^2 + b^2x.$
10. $(x - 1)(x - 2) = (x - a)^2.$
11. $a^2x = (a - b)^2 + b^2x.$
12. $(a - b)x = a^2 - (a + b)x.$
13. $\dfrac{ax}{b} + \dfrac{bx}{a} = \dfrac{a}{b} - \dfrac{b}{a}.$
14. $\dfrac{a + x}{a - 2x} = \dfrac{a - x}{a + 2x}.$
15. $\dfrac{4x - a}{2x - a} - 1 = \dfrac{x + a}{x - a}.$
16. $\dfrac{x}{a} + \dfrac{x}{b} + \dfrac{x}{c} = d.$
17. $\dfrac{3}{4}\left(\dfrac{x}{a} - 1\right) = \dfrac{2}{3}\left(\dfrac{x}{a} + 1\right).$
18. $\dfrac{2a}{3}\left(\dfrac{x}{a} - a\right) = \dfrac{3a}{5}\left(\dfrac{x}{a} + a\right).$
19. $\dfrac{ax - b}{ab} + \dfrac{bx - c}{bc} + \dfrac{cx - a}{ac} = 0.$
20. $\dfrac{ax}{3a + b} + \dfrac{bx}{3a - b} = \dfrac{3a^2x + b^2}{9a^2 - b^2}.$
21. $\dfrac{x}{a} - \dfrac{x}{a - b} = \dfrac{1}{a + b} - \dfrac{x}{b}.$
22. $\dfrac{a^2 - x}{c} - \dfrac{b^2 - x}{a} - \dfrac{c^2 - x}{b} = \dfrac{a^2}{c} - \dfrac{b^2}{a}.$
23. $\dfrac{5a^2 - 7x}{3ab} + \dfrac{ab^2 + 10x}{5ac} = \dfrac{10c^2 + 3x}{6bc} + \dfrac{5(a - c)}{3b} + \dfrac{b^2}{5c}.$

24. $\dfrac{\dfrac{a+x}{a}}{x-a} = \dfrac{1-\dfrac{x}{a}}{\dfrac{1}{a}-x}.$ **25.** $\dfrac{\dfrac{x-1}{a}}{\dfrac{a+1}{x}} = \dfrac{\dfrac{x}{a-1}}{\dfrac{a}{a+x}}.$

26. Make up and solve two literal equations.

27. How many examples in Exercise 35 (p. 131) can you now work at sight?

EXERCISE 64

ORAL

Solve the following orally, without transposing any term containing x:

1. $4x = -12$

2. $3x = a$

3. $ax = 5$

4. $ax = b$

5. $2x - 4 = 6$

6. $ax - 5 = 7$

7. $bx + c = d$

8. $ax + bx = c$

9. $ax = c + 5x$

10. $6 = 3x$

11. $10 = -5x$

12. $\dfrac{2}{3} = 3x$

13. $a = 5x$

14. $c = dx$

15. $3x = \dfrac{1}{2}$

16. $2x = -\dfrac{4}{5}$

17. $5x = \dfrac{3}{4}$

18. $ax = \dfrac{1}{b}$

19. $\dfrac{x}{3} = 2$

20. $\dfrac{x}{a} = 5$

21. $\dfrac{1}{x} = 2$

22. $\dfrac{2}{x} = 3$

23. $\dfrac{a}{x} = 1$

24. $\dfrac{-3}{x} = 9$

25. $\dfrac{111}{x} = -37$

26. $\dfrac{a}{x} = p$

27. $\dfrac{1}{3}x = 2$

28. $\dfrac{2}{5}x = 4$

29. $-\dfrac{3}{4}x = 6$

30. $\dfrac{x}{3} = \dfrac{1}{4}$

31. $4 = \dfrac{x}{2}$

32. $-3 = \dfrac{2x}{5}$

33. $4 = \dfrac{2x}{7}$

34. $a = \dfrac{bx}{c}$

35. $\dfrac{x}{2} = \dfrac{3}{5}$

36. $\dfrac{1}{2}x = \dfrac{4}{7}$

37. $\dfrac{x}{4} = -\dfrac{2}{3}$

38. $\dfrac{1}{x} = \dfrac{1}{2}$

39. $\frac{2}{x}=\frac{3}{5}$ **41.** $\frac{2}{3}=\frac{3}{x}$ **43.** $\frac{ax}{b}=\frac{c}{d}$

40. $-\frac{7}{3}=\frac{1}{x}$ **42.** $\frac{2}{3}x=\frac{1}{5}$ **44.** $\frac{4}{x}=\frac{12}{15}$

45. $p+q=\frac{a+b}{x}$ **47.** $\frac{b}{x-a}=m$

46. $\frac{c-d}{x}=p+q$ **48.** $\frac{a+b}{x}=\frac{c+d}{5}$

49. How many examples in Exercise 45 (p. 155) can you now work at sight?

EXERCISE 65

REVIEW

Solve for x and check:

1. $\frac{2}{x-2}-\frac{5}{x+2}=\frac{2}{x^2-4}.$ **2.** $\frac{x}{2}-\frac{x^2-1}{x-1}=5.$

3. $\frac{6x+1}{15}-\frac{2x-1}{5}-\frac{2x-4}{7x-16}=0.$

4. $\frac{x-b}{x-2a}-\frac{x+b}{x+2a}-\frac{4a^2-b^2}{x^2-4a^2}=0.$

5. $\frac{3px-3qx}{x^2-q^2}-\frac{p-2q}{x+q}+\frac{p-q}{q-x}=0.$

6. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

7. $6+\frac{1}{4}\left(x-9-\frac{x-3}{5}\right)+3+\frac{x-5}{5}=\frac{x}{2}.$

Find the value of x in the shortest way, when

8. $\frac{44}{7}x=\frac{44}{7}\times 19+\frac{44}{7}\times 41.$

9. $3.1416x=3.1416(723)-3.1416(476).$

10. $\frac{x}{3}-\frac{x}{4}=2.$ **11.** $\frac{2x}{3}-\frac{x}{2}=5.$

12. If $\frac{3}{x}-\frac{2}{y}=4$, find the value of x when $y=\frac{1}{3}$. Also when $y=-\frac{3}{4}$. When $y=\frac{4}{5}$.

13. Solve for l: $\frac{211}{108} = \frac{\frac{2}{3}l - \frac{3}{4}}{-\frac{1}{3}}$.

14. Solve for a: $\frac{-640}{243} = a\left(\frac{-128}{2187}\right)$.

15. In adding $\frac{x-1}{4} + \frac{5x}{6}$ we retain the L. C. D. 24. In solving the equation $\frac{x-1}{4} = \frac{5x}{6}$ and clearing of fractions, the L. C. D. 24 disappears. What is the reason for this difference?

16. Make up an example similar to Ex. 15.

17. Make up and solve an equation which contains fractions with the denominators 8, $2(x - 1)$, and 4.

18. Make up and solve an equation which contains fractions with the denominators $a + b$, $a - b$, and $b^2 - a^2$.

EXERCISE 66

1. Find the number the sum of whose third, fourth, and fifth parts is 94.

2. Make up and work a problem concerning one fourth and one sixth of some number.

3. State $\frac{x}{3} - \frac{x}{4} = 28$ as a problem concerning a number and find the number.

4. A certain number exceeds the sum of its third, fourth, and tenth parts by 38. Find the number.

5. A piece of bronze weighs 415 pounds. It contains twice as much zinc as tin, and 8 times as much copper as tin. How many pounds of each material are in the bronze?

6. Find two consecutive numbers such that one seventh of the greater exceeds one ninth of the less by 1.

7. Express in symbols 15% of x. 5% of x. 115% of b.

8. Two men kept a store for a year and made \$4800. The man who owned the store building received 40% more of the profits than the other. How much did each receive?

9. In building a macadam road the county pays twice as much as the state, and the township pays 50% more than the state. How much does each pay if the road costs \$18,000?

10. Separate \$770 into two parts so that one shall exceed the other by 20%. By $33\frac{1}{3}$%.

11. The difference of two numbers is 9. 3 increased by $\frac{5}{11}$ of the less of the two numbers equals $\frac{3}{7}$ of the greater. Find the numbers.

12. The iron ore in the United States is $\frac{1}{4}$ of the iron ore in the rest of the world. If there are 75,000,000,000 tons of iron ore in the entire world, how many tons of iron ore are there in the United States?

13. The population of India is $\frac{5}{9}$ that of China, and the population of the rest of the world is $3\frac{3}{5}$ times that of India. What is the population of India and China, if that of the entire world is 1,500,000,000?

14. A man bequeathed \$60,000 to his wife and three children. In a first will he bequeathed his wife three times as large a share as one child received. Later he changed his will and bequeathed his wife \$10,000 more than the share of a child. By which of the two wills would she have received the larger amount?

15. In one kind of concrete, the parts of cement, sand, and gravel are 1, 2, and 4. In another kind of concrete these parts are 1, 3, 5. How many more cubic feet of cement are needed to make 5600 cu. ft. of concrete of the first kind than of the second?

16. A girl's grades are, arithmetic 87, reading 92, and geography 85. What grade must she have in spelling to make her general average 90?

17. The average wheat crop of the United States for four years was 660 millions of bushels. What would the crop for the fifth year need to be in order to make the average for the five years 700 million bushels?

18. A pupil has worked 15 problems. If he should work 9 more problems and get 8 of them right, his average would be .75. How many problems has he worked correctly thus far?

19. A baseball nine has played 36 games of which it has won 25. How many games must it win in succession to bring its average of games won up to .75?

20. Make up and work an example similar to Ex. 20.

21. A baseball nine has won 19 games out of 36 games played. If after this it should win $\frac{3}{4}$ of the games played, how many games would it need to play to bring its average of games won up to $66\frac{2}{3}$?

22. A baseball nine has won 25 games out of 36 played. It still has 12 games to play. How many of these will it need to win in order to bring its average of games won up to .75?

23. How much water must be added to 50 gallons of milk containing 8% of butter fat to make a mixture containing 5% of butter fat?

SUG. The 50 gal. of milk contain 50 × .08 or 4 gal. of cream. If x denotes the number of gallons of water, $\frac{4}{50+x} = \frac{5}{100}$, etc.

24. A certain kind of cream is $\frac{4}{5}$ butter fat, and a certain kind of milk is 3% butter fat. How many gallons of the cream must be added to 40 gallons of milk to make a mixture which is 5% butter fat?

25. Of what type is each of the above problems an example or variation?

26. A mass of copper and silver alloy weighs 120 lb. and contains 8 lb. of copper. How much copper must be added to the mass in order that 100 lb. of the resulting alloy shall contain 10 lb. of the copper?

27. A mass of copper and silver alloy weighs 120 lb. and contains 8 lb. of silver. How much silver must be added to the mass in order that 1 lb. of the resulting alloy shall contain $2\frac{1}{5}$ oz. of silver?

28. If 100 lb. of sea water contains $2\frac{1}{2}$ lb. of salt, how much fresh water must be added to it in order that 100 lb. of the mixture shall contain 1 lb. of salt?

29. How much fresh water must be added to 100 lb. of sea water in order that 20 lb. of the mixture shall contain 4 oz. of salt?

30. How much water must be evaporated from 100 lb. of salt water in order that 8 lb. of the water left shall contain 1 lb. of salt?

31. How much water must be added to a gallon of alcohol which is 90% pure, in order to make a mixture which is 80% pure?

32. If it takes a man 9 days to do a piece of work, what part of it will he do in one day? If it takes him x days to do the work, what part of it will he do in one day?

33. If a boy can do a piece of work in 15 days which a man can do in 9 days, how long would it take both working together to do the piece of work?

SUG. What fractional part of the work will the boy do in 1 day? The man? If together the boy and man can do the piece of work in x days, what part of the work can they do together in 1 day?

34. A can spade a garden in 3 days, B in 4 days, and C in 6 days. How many days will they require working together?

35. A and B together can mow a field in 4 days, but A alone could do it in 12 days. In how many days can B mow it?

36. A and B in $5\frac{1}{7}$ days accomplish a piece of work which A and C can do in 6 days or B and C, in $7\frac{1}{5}$ days. If they all work together, how many days will they require to do the same work?

37. One pipe can fill a given tank in 48 min. and another can fill it in 1 h. and 12 min. How long will it take the pipes together to fill the tank?

38. Two inflowing pipes can fill a cistern in 27 and 54 min. respectively, and an outflowing pipe can empty it in 36 min. All pipes are open and the cistern is empty; in how many minutes will it be full?

SUG. Since emptying is the opposite of filling, we may consider that a pipe which empties $\frac{1}{36}$ of a cistern in a minute will *fill* $-\frac{1}{36}$ of it each minute.

39. A tank has four pipes attached, two filling and two emptying. The first two can fill it in 40 and 64 min. respectively, and the other two can empty it in 48 and 72 min. respectively. If the tank is empty and the pipes all open, in how many minutes will it be full?

40. At what time between 3 and 4 o'clock are the hands of a watch pointing in opposite directions?

SOLUTION. At 3 o'clock the minute-hand is 15 minute-spaces behind the hour-hand, and finally is 30 spaces in advance: therefore the minute-hand moves over 45 spaces more than the hour-hand.

Let x = the number of spaces the minute-hand moves
Then $x - 45$ = " " " " " hour-hand "

But the minute-hand moves 12 times as fast as the hour-hand;

hence, $x = 12(x - 45)$. Solving, $x = 49\frac{1}{11}$.

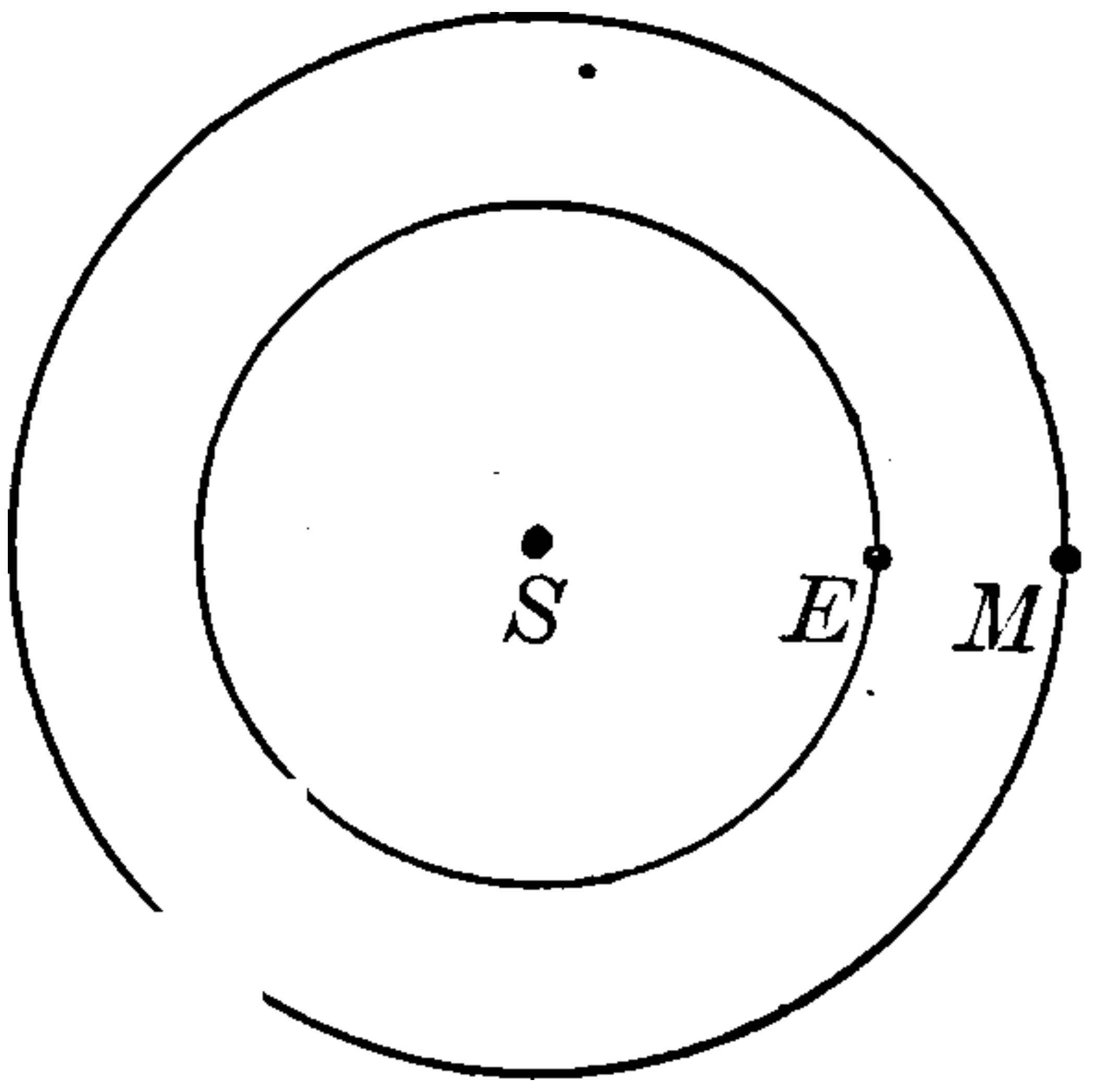
Thus the required time is $49\frac{1}{11}$ min. past 3.

41. When are the hands of a clock pointing in opposite directions between 4 and 5? Between 1 and 2?

42. What is the time when the hands of a clock are together between 6 and 7? Between 10 and 11?

43. At what instants are the hands of a watch at right angles between 4 and 5 o'clock? Between 7 and 8?

44. The planet Mars is in the most favorable position to be observed from the earth when it is in line with the earth and on opposite sides of the earth from the sun (Mars is then said to be in *opposition*). If the year is taken as 365 days, and it takes Mars 687 days to make one revolution about the sun, how long is the interval between two successive oppositions of Mars?



SUG. If it takes the earth x days to overtake Mars and thus put Mars again in opposition, how many revolutions about the sun does the earth make in x days? How many revolutions does Mars make in x days? In the interval from one opposition to the next, how many more revolutions about the sun does the earth make than Mars?

45. It takes the planet Jupiter 12 yr. to make one revolution about the sun. How long is it from one opposition of Jupiter to the next?

46. The interval between two successive oppositions of Mars is 780 days. Determine the time it takes Mars to make one revolution about the sun (i. e. the length of the year on Mars).

47. A courier travels 5 mi. an hour for 6 hours, when another courier starts at the same place and follows him at the rate of 7 mi. an hour. In how many hours will the second overtake the first?

SUG. If x = the number of hours the second courier travels, how many hours does the first courier travel? How many miles (in terms of x) does the first courier travel? The second? Do the two couriers travel equal distances?

48. A courier who travels $5\frac{1}{2}$ mi. an hour was followed after 8 hours by another, who went $7\frac{1}{3}$ mi. an hour. In how many hours will the second overtake the first?

49. A woman can write 15 words per minute with a pen, and a girl can write 40 words per minute on the typewriter. The woman has a start of 3 hours in copying a certain manuscript. How long before the girl using the typewriter will overtake the woman?

50. A train running 40 mi. an hour left a station 45 min. before a second train running 45 mi. an hour. In how many hours will the second train overtake the first?

51. A gentleman has 10 hours at his disposal. He walks out into the country at the rate of $3\frac{1}{2}$ mi. an hour and rides back at the rate of $4\frac{1}{2}$ mi. an hour. How far may he go?

52. A and B start out at the same time from P and Q, respectively, 82 mi. apart. A walked 7 mi. in 2 hours, and B 10 mi. in 3 hours. How far and how long did each walk before coming together, if they walked toward each other? If A walked toward Q, and B in the same direction from Q?

53. A certain room is 20 ft. long and 12 ft. wide. The walls and ceiling of the room together have an area of 752 sq. ft. How high is the ceiling?

54. A rifle ball is fired at a target 1100 yd. distant and $4\frac{1}{2}$ sec. after firing the shot the marksman heard the impact of the bullet on the target. If the bullet traveled at the rate of 2200 ft. per second, what was the rate at which the sound of the impact traveled back to the marksman?

55. A rifle ball is fired at a target 1000 yd. distant and 4 sec. after firing the shot, the marksman heard the impact of the bullet on the target. If sound traveled at the rate of 1100 ft. per second, at what rate did the bullet travel?

56. A 21 lb. mass of gold and silver alloy when immersed in water weighed only 19 lb. If the gold lost $\frac{1}{19}$ of its weight when weighed under water, and the silver $\frac{1}{10}$ of its weight, how many pounds of each metal were in the alloy?

SUG. If x denotes the number of pounds of gold, how many pounds of silver were there in the mass?

The law involved in the above example is that when any object is weighed in water, it loses in weight an amount equal to the weight of the water which it displaces. Hence, if the specific gravity of gold is approximately 19, the weight of the water displaced by the gold $= \frac{1}{19}$ of the weight of the gold.

Find out if you can who first used this method of determining the relative amounts of metal in an alloy and what use he first made of the method.

57. An alloy of aluminum and iron weighs 80 lb., but when immersed in water it weighs only 60 lb. If the specific gravity of aluminum is $2\frac{1}{2}$ while that of iron is $7\frac{1}{2}$, how many pounds of each metal are in the alloy?

58. A mass of copper and tin weighing 100 lb. when immersed in water weighed 87.5 lb. If the specific gravity of copper is 8.8 and that of tin is 7.3, how much of each metal was in the mass?

59. If a bushel of oats is worth 40¢ and a bushel of corn is worth 55¢, how many bushels of each grain must a miller use to produce a mixture of 100 bu. worth 48¢ a bushel?

60. A man has $5050 invested, some at 4%, and some at 5%. How much has he at each rate if the annual income is $220?

61. Divide the number 54 into 4 parts, such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, will all produce equal results.

62. Find three consecutive numbers such that if they be divided by 2, 3, and 4 respectively, the sum of the quotients will equal the next higher consecutive number.

63. In the United States the gold dollar is 90% gold and 10% copper. If a mass of gold and copper weighing 24 lb. is 75% gold, how many pounds of gold must be added to it to make it ready for coinage into gold dollars?

64. My annual income is $990. If $\frac{1}{4}$ of my property is invested at 5%, $\frac{2}{5}$ at 4%, and the rest at 6%, find the amount of my property.

65. If one pipe can fill a swimming tank in 1 hour and another can fill it in 36 minutes, how long will it take the two pipes together to fill the tank?

66. At what time are the hands of a watch at right angles between 10 and 11 o'clock?

67. If one baseball nine has won 16 games out of 42 played, and another has won 18 out of 40 played, how many straight games must the first team win in order at least to equal the average of games won by the second team?

68. If the interval between two successive oppositions of the planet Saturn is 378 days, how long is the year on Saturn?

69. If A, B, and C together can do in $5\frac{1}{3}$ days a certain amount of work, which B alone could do in 24 days, or C alone in 16 days, how long would A require?

70. How much water must be added to 1 gal. of a 5% solution of a certain chemical to reduce it to a 2% solution?

71. A baseball player who has been at the bat 150 times has a batting average of .240. How many more times must he bat in order to bring his average up to .250, provided that in the future his base hits equal half the number of times he bats?

72. A girl has worked a certain number of problems and has $\frac{2}{3}$ of them right. If she should work 9 more problems and get 8 of them right, her average would be .75. How many problems has she worked?

73. If the sum of two consecutive integers is $4p + 5$, find the integers.

74. A man has a hours at his disposal. He wishes to ride out into the country and walk back. How far may he ride in a coach which travels b miles an hour, and return home in time, walking c miles an hour?

75. Generalize Ex. 33; that is, make up and work a similar example where letters are used instead of figures for the known numbers.

76. If E denotes the number of days it takes the earth to revolve once around the sun, P denotes the number of days it takes a planet (as Mars) to complete a revolution about the sun, and S the number of days between two successive oppositions of the planet, show that $\frac{1}{E} - \frac{1}{P} = \frac{1}{S}$.

77. The fore wheel of a carriage is a feet in circumference and the hind wheel is b feet. What distance has been passed over when the fore wheel has made c revolutions more than the hind wheel?

78. Make up and work three examples similar to such of the examples in this Exercise as the teacher may point out.

EXERCISE 67

1. Given $V = lwh$, find h in terms of the other letters. Also solve for l. For w.

2. Given $i = prt$, find each letter in terms of the others.

Find each letter in terms of the others in the following formulas used in geometry:

3. $K = \frac{1}{2}bh$

4. $K = \frac{1}{2}h(b + b')$

5. $C = 2\pi R$

6. $S = \pi RL$

7. $T = \pi R(R + L)$

8. $T = 2\pi R(R + H)$

Also find each letter in terms of the others in the following formulas used in mechanics and physics:

9. $S = vt$

10. $LW = lw$

11. $C = \dfrac{5}{9}(F - 32)$

12. $C = \dfrac{E}{R}$

13. $R = \dfrac{gs}{g + s}$

14. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{p}$

15. By use of the formula in Ex. 2 determine in how many years $325 will produce $84.50 interest at 5 per cent.

16. Also find the rate at which $176 will yield $43.56 interest in 5 yr. 6 mo.

17. Change the following temperatures on the Centigrade scale to Fahrenheit readings:

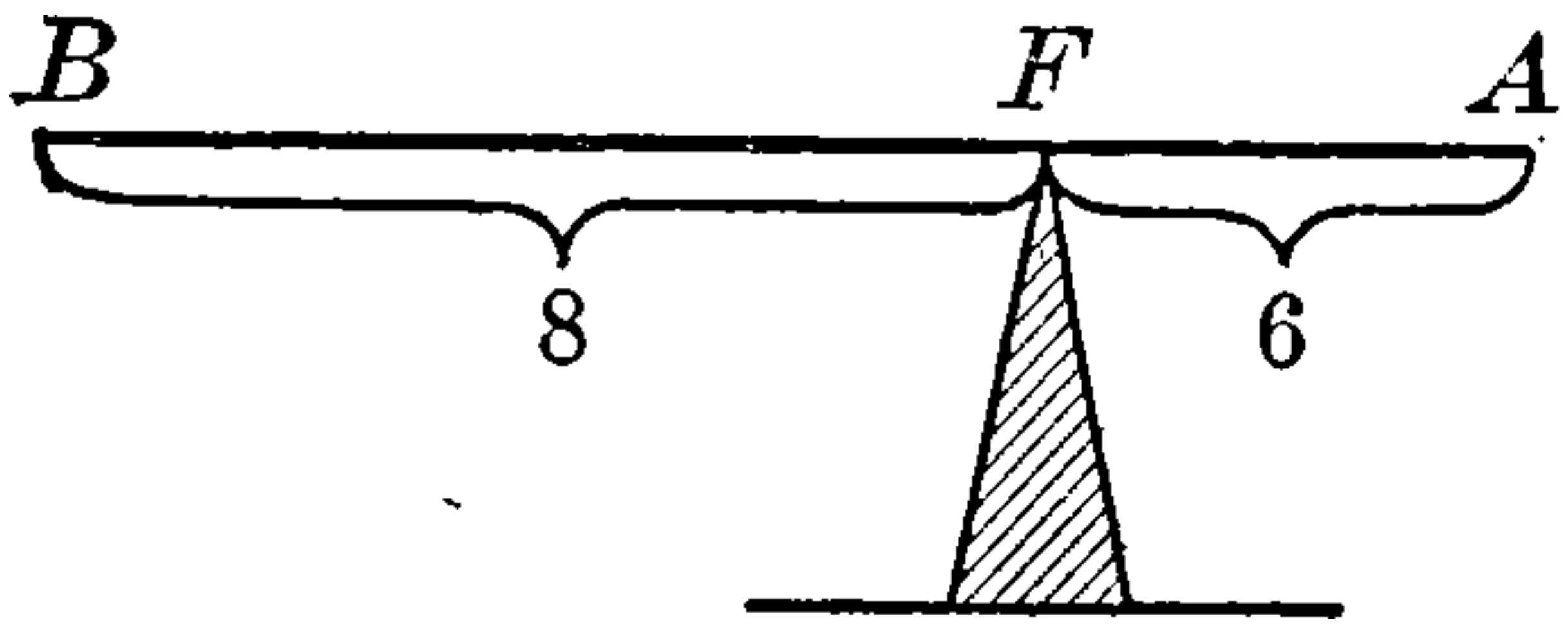
(1) 50° (2) 0° (3) 2700°

18. Metals fuse at the following temperatures on the Centigrade scale. What are the temperatures at which they fuse on the Fahrenheit scale?

Tin 228° Lead 325° Copper 1091° Iron 1540°

19. Solve the following equation for b: $\dfrac{\dfrac{bc + d}{d}}{\dfrac{bc}{d}} = \dfrac{2d}{a}$.
Also solve for c. For d.

20. A boy who weighs 80 lb. is on a teeter board at A, 6 ft. from the fulcrum F. He just balances a boy who is at B on the same board, 8 ft. from F. What does the second boy weigh? (Use the formula of Ex. 10.)



21. Make up and work an example similar to Ex. 19.

22. How many examples in Exercise 48 (p. 163) can you now work at sight?

CHAPTER XII

SIMULTANEOUS EQUATIONS

132. Need and Utility of Simultaneous Equations.

Ex. A farmer one year made a profit of \$2221 on 27 acres of corn and 40 acres of potatoes. The next year with equally good crops, he made a profit of \$2028 on 36 acres of corn and 30 acres of potatoes. How much did he make per acre on his corn and on his potatoes?

Let x = no. of dollars made on 1 acre of corn
y = " " " " " 1 " " potatoes

$$\text{Then } 27x + 40y = 2221$$
$$36x + 30y = 2028$$

From these equations the value of x may be found by combining the equations in some way which will get rid of, or eliminate, y. (See Arts. 136–138.)

Try to solve the above problem by the use of only one unknown, as x. If you succeed at all, you will find the method awkward and inconvenient.

Why do we now proceed to make definitions and rules?

133. Simultaneous Equations are a set or system of equations in which more than one unknown quantity is used, and the same symbol stands for the same unknown number.

Thus, in the group of three simultaneous equations,

$$x + y + 2z = 13$$
$$x - 2y + z = 0$$
$$2x + y - z = 3$$

x stands for the same unknown number in all of the three equations, y for another unknown number, and z for still another.

134. Independent Equations are those which cannot be derived one from the other.

Thus, $x + y = 10,$
and $2x = 20 - 2y,$

are not independent equations, since by transposing $2y$ in the second equation and dividing it by 2, we may convert the second equation into the first.

But $\left.\begin{aligned} 3x - 2y &= 5 \\ 5x + y &= 6 \end{aligned}\right\}$ are independent equations, since neither one of them can be converted into the other.

135. Elimination is the process of combining two equations containing two unknown quantities so as to form a single equation with only one unknown quantity. Or, in general, elimination is the process of combining several simultaneous equations so as to form equations one less in number and containing one less unknown quantity.

There are three principal methods of elimination: I, *addition and subtraction*; II, *substitution*; and III, *comparison*.

These methods are presented to best advantage in connection with illustrative examples.

136. I. Elimination by Addition and Subtraction.

Ex. Solve
$$\begin{cases} 12x + 5y = 75 \quad \dots\dots\dots\dots (1) \\ 9x - 4y = 33 \quad \dots\dots\dots\dots (2) \end{cases}$$

In order to make the coefficients of y in the two equations alike, we multiply equation (1) by 4, and (2) by 5,

$$48x + 20y = 300 \quad \dots\dots\dots\dots (3)$$
$$45x - 20y = 165 \quad \dots\dots\dots\dots (4)$$

Add equations (3) and (4), $93x = 465$
Divide by 93, $x = 5$ *Root*

Substitute for x its value 5, in equation (1),

$$60 + 5y = 75$$
$$\therefore y = 3 \text{ } \textit{Root}$$

CHECK. $12x + 5y = 12 \times 5 + 5 \times 3 = 75$
$9x - 4y = 9 \times 5 - 4 \times 3 = 33$

Since y was eliminated by adding equations (3) and (4), the above process is called elimination by *addition.*

The same example might have been solved by the method of subtraction.

Thus, multiply equation (1) by 3, and (2) by 4,

$$36x + 15y = 225 \qquad (5)$$

$$36x - 16y = 132 \qquad (6)$$

Subtract (6) from (5), $31y = 93$

$$y = 3$$

and $x = 5$

It is important to select, in every case, the smallest multipliers that will cause one of the unknown quantities to have the same coefficient in both equations.

Thus, in the last solution given above, instead of multiplying equation (1) by 9, and (2) by 12, we divide these multipliers by their common factor, 3, and get the smaller multipliers, 3 and 4.

Hence, in general,

Multiply the given equations by the smallest numbers that will cause one of the unknown quantities to have the same coefficient in both equations;

If the equal coefficients have the same sign, subtract the corresponding members of the two equations; if the equal coefficients have unlike signs, add.

EXERCISE 68

Solve by addition and subtraction:

1. $3x - 2y = 1$
$x + y = 2$

2. $2x - 7y = 9$
$5x + 3y = 2$

3. $4x + 3y = 1$
$2x - 6y = 3$

4. $5x - 3y = 1$
$3x + 5y = 21$

5. $x + 5y = -3$
$7x + 8y = 6$

6. $3x - 2y = 4$
$5x - 4y = 7$

7. $$\begin{aligned} 2y + x &= 0 \\ 4x + 6y &= -3 \end{aligned}$$

8. $$\begin{aligned} 9x - 8y &= 5 \\ 15x + 12y &= 2 \end{aligned}$$

9. $$\begin{aligned} 4x - 6y + 1 &= 0 \\ 5x - 7y + 1 &= 0 \end{aligned}$$

10. $$\begin{aligned} 8x + 5y &= 6 \\ 6y + 2x &= 11 \end{aligned}$$

11. $$\begin{aligned} 5x - 3y &= 36 \\ 7x - 5y &= 56 \end{aligned}$$

12. $$\begin{aligned} \frac{x}{2} - \frac{y}{3} &= 1 \\ \frac{x}{4} - \frac{y}{9} &= 1 \end{aligned}$$

13. $$\begin{aligned} \frac{x}{3} - \frac{y}{5} &= 3 \\ \frac{x}{5} + \frac{y}{2} &= 8 \end{aligned}$$

14. $$\begin{aligned} \frac{2x}{3} + \frac{y}{4} &= 1 \\ \frac{3x}{2} + \frac{5y}{8} &= 2 \end{aligned}$$

15. $$\begin{aligned} \frac{4x}{5} + \frac{3y}{2} &= -7 \\ \frac{3x}{4} + \frac{2y}{5} &= \frac{7}{2} \end{aligned}$$

16. $$\begin{aligned} \frac{5x}{6} - \frac{8y}{9} &= -6 \\ \frac{3x}{4} - \frac{5y}{6} &= -6 \end{aligned}$$

17. Find two numbers whose sum is 12 and whose difference is 2.

18. The half of one number plus the third of another number equals 13, while the sum of the numbers is 33. Find the numbers.

19. State Ex. 1 as a problem concerning two numbers.

20. State Ex. 2 as a problem concerning two numbers.

21. 7 lb. of sugar and 3 lb. of rice together cost 57¢; also 5 lb. of sugar and 6 lb. of rice cost 60¢. Find the cost of a pound of each.

22. Make up and work an example similar to Ex. 18. To Ex. 21.

23. How many examples in Exercise 50 (p. 170) can you now work at sight?

137. II. Elimination by Substitution.

Ex. Solve $5x + 2y = 36 \quad \ldots\ldots\ldots\ldots \quad (1)$

$2x + 3y = 43 \quad \ldots\ldots\ldots\ldots \quad (2)$

From (1) $5x = 36 - 2y$

$$\therefore \quad x = \frac{36 - 2y}{5} \quad \ldots\ldots\ldots\ldots \quad (3)$$

In equation (2) substitute for x its value given in (3),

$$2\left(\frac{36 - 2y}{5}\right) + 3y = 43$$

$$\frac{72 - 4y}{5} + 3y = 43$$

$$72 - 4y + 15y = 215$$

$$11y = 143$$

$$y = 13 \text{ \textit{Root}}$$

Substitute for y in (3), $x = \dfrac{36 - 26}{5} = 2$ *Root*

Let the pupil check the work.

Hence, in general,

In one of the given equations obtain the value of one of the unknown quantities in terms of the other unknown quantity;

Substitute this value in the other equation and solve.

EXERCISE 69

1. Work the examples of Exercise 68 (p. 225) by the method of substitution.

Find out which of the following sets of equations are worked more readily by the method of addition and subtraction, and which by the method of substitution, and work each example accordingly:

2. $x = 3y - 5$
$2x + 5y = 12$

3. $3x - 4y = 1$
$4x - 5y = 1$

4. $x - 3 = 0$
$2y + 3x = 5$

5. $2x + 3y = 1$
$3x + 4y = 2$

6. $7x + 8y = 19$
$5x + 6y = 13\frac{1}{2}$

7. $x = 2y - 3$
$y = 5x - 21$

8. $y = 3$
$2x = 3y - 17$

9. $y = 3x$
$4x + 5y = 38$

10. Make up and solve an example in simultaneous equations which is solved more readily by the method of addition and subtraction than by the method of substitution.

11. Make up and solve an example of which the reverse of Ex. 10 is true.

12. How many examples in Exercise 51 (p. 172) can you now work at sight?

138. III. Elimination by Comparison.

Ex. Solve $2x - 3y = 23 \ldots\ldots (1)$

$5x + 2y = 29 \ldots\ldots (2)$

From (1) $2x = 23 + 3y \ldots\ldots (3)$

From (2) $5x = 29 - 2y \ldots\ldots (4)$

From (3) $x = \frac{23 + 3y}{2} \ldots\ldots (5)$

From (4) $x = \frac{29 - 2y}{5} \ldots\ldots (6)$

Equate the two values of x in (5) and (6),

$$\frac{23 + 3y}{2} = \frac{29 - 2y}{5}$$

Hence, $115 + 15y = 58 - 4y$

$$19y = -57$$

$$y = -3 \text{ Root}$$

Substitute for y in (5), $x = \frac{23 - 9}{2} = 7$ *Root*

Let the pupil check the solution.

Hence, in general,

Select one unknown quantity, and find its value in terms of the other unknown quantity in each of the given equations;

Equate these two values, and solve the resulting equation.

EXERCISE 70

1. Work the examples of Exercise 68 (p. 225) by the method of comparison.

Ascertain by which of the three methods of elimination each of the following examples can be worked most readily, and solve accordingly:

2. $x = 3y + 9$
$x = 5y + 13$

3. $x = 3y + 9$
$3x - 5y = 10$

4. $6x + 5y - 8 = 0$
$4x - 3y - 18 = 0$

5. $y = 2x$
$3x + 2y = 21$

6. $y = 6x - 3$
$8 - 5x = y$

7. $9x + 12y = -6$
$6x - 5y = -17$

8. $x = 5$
$3x - 2y = 13$

9. $5x + 3y = 8$
$5x - 4y = 7$

10. $y = \frac{2}{5}(x - 3)$
$y = \frac{3}{4}x + 1$

11. $y = 2x + 1$
$3x + \dfrac{5y}{2} = 8$

12. Make up and solve an example in simultaneous equations which is solved more readily by the method of comparison than by either of the other two methods of elimination.

13. Make up and solve an example in simultaneous equations which is solved more readily by the method of substitution than by either of the other two methods.

14. Make up and solve an example solved more readily by the method of addition and subtraction than by the other two methods.

EXERCISE 71

Solve and check each result:

1. $\dfrac{x+1}{3} - \dfrac{y}{4} = -1$

$\dfrac{y}{3} - \dfrac{3x+1}{4} = 0$

2. $x - \dfrac{y-2}{7} = 5$

$4y - \dfrac{x+10}{3} = 3$

3. $\dfrac{4x+5y}{40} = x - y$

$\dfrac{2x-y}{3} + 2y = \frac{1}{2}$

4. $\dfrac{5x+y}{13} - \dfrac{2x+y}{11} = -1$

$3y - x = 2$

5. $\dfrac{2x-y}{5} + \dfrac{3x+2y}{11} = 2$

$-\dfrac{2x}{3} + \dfrac{4x+y-1}{4} = 1$

6. $\dfrac{7}{y+3} - \dfrac{3}{x+4} = 0$

$y(x-2) - x(y-5) + 13 = 0$

7. $\frac{2}{3}(x+3y) - \frac{1}{2}(x+2y) = \frac{7}{12}$

$3y - \frac{2}{3}(x + 4y + \frac{5}{6}) = 0$

8. $.4x - .3y = .7$

$.7x + .2y = .5$

9. $2x + 1.5\,y = 10$

$.3x - .05y = .4$

10. $.5x + 5.5y = 2.6$

$1.3x + 3.1y = 1.6$

11. $.8x - .7y = .005$

$2x = 3y$

12. $\dfrac{1-3y}{x + \dfrac{2-3x}{3}} = x + 13$

$\dfrac{10y+1}{5} = y - \dfrac{2x+3}{x + 3 - \dfrac{1+2x}{2}}$

13. $$\frac{x-y}{x+y}=\frac{1}{5}$$
$$\frac{\frac{y}{3}-\frac{3x}{2}}{11\frac{1}{2}}-\frac{\frac{5y}{12}-\frac{2x}{3}}{1\frac{3}{4}}=2$$

14. $$(x-5)(y+3)=(x-1)(y+2)$$
$$xy+2x=x(y+10)+72y$$

15. $$\frac{x-2}{5}+\frac{x+10}{3}+\frac{10-y}{4}=13$$
$$\frac{2y+6}{3}-\frac{4x+y+6}{8}+4=0$$

16. $$\frac{6y+5}{8}-\frac{3x+5\frac{1}{2}}{5x-2y}=\frac{9y-4}{12}$$
$$\frac{2y+3}{4}+\frac{x+y}{3x-2y}=\frac{4y+7}{8}$$

17. $$\frac{3x-2}{5}=\frac{6x-5}{10}-\frac{x+y+6\frac{1}{2}}{6x+y}$$
$$\frac{3y-2}{12}=\frac{2y-5}{8}+\frac{3+7x}{10y-3x}$$

18. Practice oral work with small fractions as in Exercise 58 (p. 190).

139. Literal Equations.

Ex. Solve
$$ax+by=c \quad \dots\dots\dots\dots (1)$$
$$a'x+b'y=c' \quad \dots\dots\dots\dots (2)$$

Multiply (1) by a', and (2) by a,
$$aa'x+a'by=a'c \quad \dots\dots\dots\dots (3)$$
$$aa'x+ab'y=ac' \quad \dots\dots\dots\dots (4)$$

Subtract (4) from (3), $(a'b-ab')y=a'c-ac'$
$$\therefore y=\frac{a'c-ac'}{a'b-ab'} \text{ Root}$$

Again, multiply (1) by b', (2) by b,

$$ab'x + bb'y = b'c \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$a'bx + bb'y = bc' \quad \ldots\ldots\ldots\ldots \quad (6)$$

Subtract (6) from (5), $(ab' - a'b)x = b'c - bc'$

$$\therefore x = \frac{b'c - bc'}{ab' - a'b} \text{ Root}$$

Let the pupil check the work.

In solving simultaneous literal equations, observe that if the value obtained for the first unknown is a fraction containing a binomial term (or the value is complex in other ways), it is better not to find the value of the other unknown as in numerical equations, i. e. by substituting the value found in one of the original equations and reducing. A better method is to take both of the original equations and eliminate anew. See the solution of the preceding example.

EXERCISE 72

Solve and check each result:

1. $3x + 4y = 2a$
$5x + 6y = 4a$

2. $2ax + 3by = 4ab$
$5ax + 4by = 3ab$

3. $ax + by = 1$
$a'x + b'y = 1$

4. $x - y = 2n$
$mx - ny = m^2 + n^2$

5. $2bx + ay = 4b + a$
$abx - 2aby = 4b + a$

6. $ax - by = a^2 + b^2$
$bx + ay = 2(a^2 + b^2)$

7. $ax + by = c$
$mx + ny = d$

8. $bx + ay = a + b$
$ab(x - y) = a^2 - b^2$

9. $c^2x - d^2y = c - d$
$cd(2dx - cy) = 2d^2 - c^2$

10. $\dfrac{x + m}{y - n} = \dfrac{n}{m}$
$x + y = 2n$

11. $(a + 1)x - by = a + 2$
$(a - 1)x + 3by = 9a$

12. $(a - b)x - (a + b)y = a^2 + b^2$
$bx + ay = 0$

13. $\dfrac{x}{a + b} + \dfrac{y}{a - b} = 2$
$x - y = 2b$

14. $ax - bx = ay - dy$
$x - y = 1$

15. $\dfrac{(a - b)x + (a + b)y}{a^2 + b^2} = 1$
$ax - 2by = a^2 - 2b^2$

16. $(a + b)x + cy = 1$
$cx + (a + b)y = 1$

17. $(a+b)x-(a-b)y=3ab$
$(a-b)x-(a+b)y=ab$

18. $\dfrac{x-b}{a+b}+\dfrac{y-b}{a-b}=-1$
$\dfrac{x+2a}{a-b}+\dfrac{y-2b}{a+b}=\dfrac{a^2+b^2}{a^2-b^2}$

19. $\dfrac{x-1}{b-1}+\dfrac{y-a}{b-a}=1$
$\dfrac{x+1}{b}+\dfrac{y-1}{1-a}=\dfrac{1}{b}$

20. $(x-1)(a+b)=a(y+a+1)$
$(y+1)(a-b)=b(x-b-1)$

21. Make up and work an example similar to Ex. 7. To Ex. 11.

140. Three or More Simultaneous Equations.

Ex. Solve
$$\begin{cases} 3x+4y-5z=32 & \ldots\ldots\ (1)\\ 4x-5y+3z=18 & \ldots\ldots\ (2)\\ 5x-3y-4z=2 & \ldots\ldots\ (3)\end{cases}$$

If we choose to eliminate z first, multiply (1) by 3, and (2) by 5,
$$9x+12y-15z=96 \quad\ldots\ldots\ (4)$$
$$20x-25y+15z=90 \quad\ldots\ldots\ (5)$$
Add (4) and (5), $$29x-13y=186 \quad\ldots\ldots\ (6)$$
Also multiply (2) by 4, (3) by 3,
$$16x-20y+12z=72 \quad\ldots\ldots\ (7)$$
$$15x-9y-12z=6 \quad\ldots\ldots\ (8)$$
Add (7) and (8), $$31x-29y=78 \quad\ldots\ldots\ (9)$$
We now have the pair of simultaneous equations,
$$\begin{cases}29x-13y=186\\31x-29y=78\end{cases}$$
Solving these, obtain $$\left.\begin{matrix}x=10\\y=8\end{matrix}\right\}\ \textit{Roots}$$
Substitute for x and y in equation (1),
$$30+32-5z=32,$$
$$z=6\quad \textit{Root}$$

CHECK. $3x+4y-5z=3\times10+4\times8-5\times6=32$
$4x-5y+3z=4\times10-5\times8+3\times6=18$
$5x-3y-4z=5\times10-3\times8-4\times6=2$

In like manner, if we have n simultaneous equations containing n unknown quantities, by taking different pairs of the n equations, we may eliminate one of the unknown quantities, leaving $n - 1$ equations, with $n - 1$ unknown quantities; and so on.

EXERCISE 73

Solve and check:

1.
$$\begin{gathered} x + y + z = 6 \\ 3x + 2y + z = 10 \\ 3x + y + 3z = 14 \end{gathered}$$

2.
$$\begin{gathered} 3x - y - 2z = 11 \\ 4x - 2y + z = -2 \\ 6x - y + 3z = -3 \end{gathered}$$

3.
$$\begin{gathered} 5x - 6y + 2z = 5 \\ 8x + 4y - 5z = 5 \\ 9x + 5y - 6z = 5 \end{gathered}$$

4.
$$\begin{gathered} 3x - \tfrac{1}{4}y + z = 7\tfrac{1}{2} \\ 2x - \tfrac{1}{3}(y - 3z) = 5\tfrac{1}{3} \\ 2x - \tfrac{1}{2}y + 4z = 11 \end{gathered}$$

5.
$$\begin{gathered} 2x + 3y = 7 \\ 3y + 4z = 9 \\ 5x + 6z = 15 \end{gathered}$$

6.
$$\begin{gathered} 2x + 4y + 3z = 6 \\ 6y - 3x + 2z = 7 \\ 3x - 8y - 7z = 6 \end{gathered}$$

7.
$$\begin{gathered} x + 3y + 3z = 1 \\ 3x - 5z = 1 \\ 9y + 10z + 3x = 1 \end{gathered}$$

8.
$$\begin{gathered} u + v - w = 4 \\ u + v - x = 1 \\ v + w + x = 8 \\ u - w + x = 5 \end{gathered}$$

9.
$$\begin{gathered} \tfrac{1}{3}x + \tfrac{1}{9}y + \tfrac{1}{6}z = 2 \\ \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{8}z = 9 \\ \tfrac{1}{6}x + \tfrac{1}{2}y + \tfrac{1}{3}z = 3 \end{gathered}$$

10.
$$\begin{gathered} 2x + 2y - z = 2a \\ 3x - y - z = 4b \\ 5x + 3y - 3z = 2(a + b) \end{gathered}$$

11.
$$\begin{gathered} \frac{2x}{3} + \frac{3y}{4} - \frac{4z}{5} = 18 \\ \frac{5x}{6} - \frac{5y}{8} + \frac{3z}{4} = -5 \\ \frac{3x}{2} - \frac{7y}{5} + \frac{3z}{10} = -41 \end{gathered}$$

12.
$$\begin{gathered} x + y + 2z = 2(a + b) \\ x + z + 2y = 2(a + c) \\ y + z + 2x = 2(b + c) \end{gathered}$$

13.
$$\begin{gathered} x + y - z = 3 - a - b \\ x + z - y = 3a - b - 1 \\ y + z - x = 3b - a - 1 \end{gathered}$$

14.
$$\begin{gathered} 3x + 2y = \tfrac{13}{6}a \\ 6z - 2x = \tfrac{5}{3}b \\ 5y - 13z + x = 0 \end{gathered}$$

15.
$$\begin{gathered} -x + y + z + v = a \\ x - y + z + v = b \\ x + y - z + v = c \\ x + y + z - v = d \end{gathered}$$

16. Practice the oral solution of simple equations as in Exercise 64 (p. 209).

141. The Use of $\frac{1}{x}$ and $\frac{1}{y}$ as Unknown Quantities enables us to solve certain equations which would otherwise be difficult of solution.

Ex. 1. Solve
$$\begin{cases} \dfrac{5}{x}+\dfrac{13}{y}=49 & \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ \dfrac{7}{x}+\dfrac{3}{y}=23 & \ldots\ldots\ldots\ldots\ldots\ldots(2) \end{cases}$$

Multiply (1) by 7, and (2) by 5,
$$\begin{cases} \dfrac{35}{x}+\dfrac{91}{y}=343 & \ldots\ldots\ldots\ldots\ldots\ldots(3) \\ \dfrac{35}{x}+\dfrac{15}{y}=115 & \ldots\ldots\ldots\ldots\ldots\ldots(4) \end{cases}$$

Subtract (4) from (3), $\frac{76}{y}=228$, $\therefore\ \frac{1}{y}=3$, or $y=\frac{1}{3}$ *Root*

Substitute the value of y in (2), hence, $x=\frac{1}{2}$ *Root*

Let the pupil check the work.

Ex. 2. Solve
$$\begin{cases} \dfrac{3}{2x}+\dfrac{5}{3y}=11 & \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ \dfrac{2}{x}-\dfrac{1}{4y}=\dfrac{29}{4} & \ldots\ldots\ldots\ldots\ldots\ldots(2) \end{cases}$$

When x and y in the denominators have coefficients, as in this example, it is usually best first to remove these coefficients by multiplying each equation by the L. C. M. of the coefficients of x and y in the denominators of that equation. Hence,

Multiply (1) by 6, and (2) by 4,
$$\begin{cases} \dfrac{9}{x}+\dfrac{10}{y}=66 & \ldots\ldots\ldots\ldots\ldots\ldots(3) \\ \dfrac{8}{x}-\dfrac{1}{y}=29 & \ldots\ldots\ldots\ldots\ldots\ldots(4) \end{cases}$$

Solving (3) and (4) by the method used in Ex. 1,
$$\left.\begin{aligned} \therefore x &= \tfrac{1}{4} \\ y &= \tfrac{1}{3} \end{aligned}\right\}\ \textit{Roots}$$

Let the pupil check the work.

EXERCISE 74

Solve and check:

1. $\frac{4}{x}+\frac{7}{y}=3$

$\frac{3}{x}+\frac{5}{y}=2$

2. $\frac{1}{x}-\frac{2}{y}=7$

$\frac{3}{x}+\frac{4}{y}=1$

3. $\frac{2}{3x}+\frac{1}{2y}=9$

$\frac{5}{6x}+\frac{4}{5y}=13$

4. $\frac{1}{2x}+\frac{1}{3y}=1$

$\frac{2}{3x}+\frac{3}{2y}=-5$

5. $\frac{3}{4x}-\frac{5}{3y}=11\frac{1}{2}$

$\frac{5}{8x}-\frac{3}{2y}=10\frac{1}{4}$

6. $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$

$\frac{1}{x}-\frac{1}{y}=n$

7. $\frac{a}{x}+\frac{b}{y}=\frac{a-b}{a}$

$\frac{b}{x}+\frac{a}{y}=\frac{b-a}{a}$

8. $\frac{m}{x}+\frac{1}{y}=m^2+n$

$\frac{1}{x}+\frac{n}{y}=m+n^2$

9. $\frac{a}{bx}+\frac{b}{ay}=2$

$\frac{b}{x}+\frac{a}{y}=\frac{a^3+b^3}{ab}$

10. $\frac{a+b}{x}+\frac{a-b}{y}=2a$

$\frac{a}{x}+\frac{b}{y}=a+b$

11. $5y-3x=7xy$

$15x+60y=16xy$

12. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$

$\frac{2}{x}-\frac{1}{y}+\frac{1}{z}=7$

$\frac{3}{x}+\frac{2}{y}+\frac{5}{z}=14$

13. $\frac{3}{x}-\frac{1}{y}=3\frac{1}{2}$

$\frac{5}{y}+\frac{3}{z}=-7$

$\frac{2}{x}-\frac{1}{z}=0$

14. $$\frac{2}{3x}-\frac{1}{2y}+\frac{4}{5z}=3\tfrac{1}{30}$$
$$\frac{3}{x}+\frac{4}{y}-\frac{3}{2z}=12$$
$$\frac{5}{2x}-\frac{3}{4y}-\frac{1}{3z}=1\tfrac{5}{12}$$

15. $$\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=\frac{1}{a}$$
$$\frac{1}{y}-\frac{1}{z}-\frac{1}{x}=\frac{1}{b}$$
$$\frac{1}{z}-\frac{1}{x}-\frac{1}{y}=\frac{1}{c}$$

16. $$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=l$$
$$\frac{a}{x}+\frac{c}{z}-\frac{b}{y}=m$$
$$\frac{b}{y}+\frac{c}{z}-\frac{a}{x}=n$$

17. $$5yz+6xz-3xy=8xyz$$
$$4yz-9xz+xy=19xyz$$
$$yz-12xz-2xy=9xyz$$

18. Make up and work an example similar to Ex. 1. To Ex. 4. Ex. 13. Ex. 15.

19. Work again such examples on pp. 212 and 213 as the teacher may point out.

142. In the **Solution of Problems Involving Two or More Unknown Quantities,** it is necessary to *obtain as many independent equations as there are unknown quantities involved in the equations* and to eliminate. (See Art. 134, p. 224.)

Ex. Find a fraction such that if 2 is added to both numerator and denominator, the fraction becomes $\frac{1}{2}$; but if 7 is added to both numerator and denominator, the fraction becomes $\frac{2}{3}$.

Two unknown numbers occur in this problem, *viz.*: the numerator and denominator of the required fraction. Hence two equations must be formed in order to obtain a solution of the problem.

Let $\frac{x}{y}$ represent the fraction.

Then, $\frac{x+2}{y+2} = \frac{1}{2}$ and $\frac{x+7}{y+7} = \frac{2}{3}$

Clearing these equations, and collecting like terms,

$$2x - y = -2$$
$$3x - 2y = -7$$

The solution gives $x = 3$ and $y = 8$.
Therefore $\frac{3}{8}$ is the required fraction.
Let the pupil check the work.

EXERCISE 75

1. Find two numbers whose sum is 23 and whose difference is 5.

2. Twice the difference of two numbers is 6, and $\frac{1}{6}$ of their sum is $3\frac{1}{2}$. What are the numbers?

3. Find two numbers such that twice the greater number exceeds 5 times the less by 6; but the sum of the greater number and twice the less is 12.

4. 2 lb. of flour and 5 lb. of sugar cost 31 cents, and 5 lb. of flour and 3 lb. of sugar cost 30 cents. Find the value of a pound of each.

5. A man hired 4 men and 3 boys for a day for $18; and for another day, at the same rate, 3 men and 4 boys for $17. How much did he pay each man and each boy per day?

6. In an orchard of 100 trees, the apple trees are 5 more than $\frac{2}{3}$ of the number of pear trees. How many trees are there of each kind?

7. One woman buys 4 yd. of silk and 7 yd. of satin, and another woman at the same rate buys 5 yd. of silk and $5\frac{4}{5}$ yd. of satin. Each woman pays $17.70. What is the price of a yard of each material?

8. Solve Ex. 7 without using x and y to represent unknown numbers (see Art. 1). About how much of the labor of writing out the solution is saved by the use of x and y?

9. 1 cu. ft. of iron and 1 cu. ft. of lead together weigh 1180 lb.; also the weight of 3 cu. ft. of iron exceeds the weight of 2 cu. ft. of lead by 40 lb. What is the weight of 1 cu. ft. of each of these materials?

10. In an athletic meet, the winning team had a score of 26 points and the second team had a score of 21 points. If the winning team took first place in 7 events and second place in 5 events, while the second team took 6 firsts and 3 seconds, how many points does a first place count? A second place?

11. In an athletic meet, the three winning teams made scores as follows:

Team	1sts	2ds	3ds	Total Score
A	5	2	2	33
B	3	3	1	25
C	1	4	6	23

What did each of the first three places in an event count in this meet?

12. Make up and work an example similar to Ex. 10.

13. Two partners agree to divide their profits each year in such a way that one partner receives $1000 more than $\frac{2}{3}$ of what the other receives. If the profits for a given year are $10,000, what does each partner receive?

14. Separate 240 into two parts such that twice the larger part exceeds five times the smaller by 10.

15. If the cost of a telegram of 14 words between two cities is 62¢, and one of 17 words is 71¢, what is the charge for the first 10 words in a message and for each word after that?

16. Make up and work an example similar to Ex. 15 concerning telegraph rates between two cities near your home.

17. A farmer one year made a profit of $1640 on 20 acres planted with wheat and 30 acres planted with potatoes. The next year, with equally good crops, he made a profit of $1210 on 30 acres planted with wheat and 20 acres planted with potatoes. How much per acre on the average did he make on each crop?

18. In three successive years, the farmer raised crops with profits as follows:

(1) 20 A. wheat, 30 A. corn, 40 A. potatoes; profits $1720
(2) 30 A. wheat, 40 A. corn, 20 A. potatoes; profits $1520
(3) 40 A. wheat, 20 A. corn, 30 A. potatoes; profits $1440

What were his average profits per acre for each kind of crop?

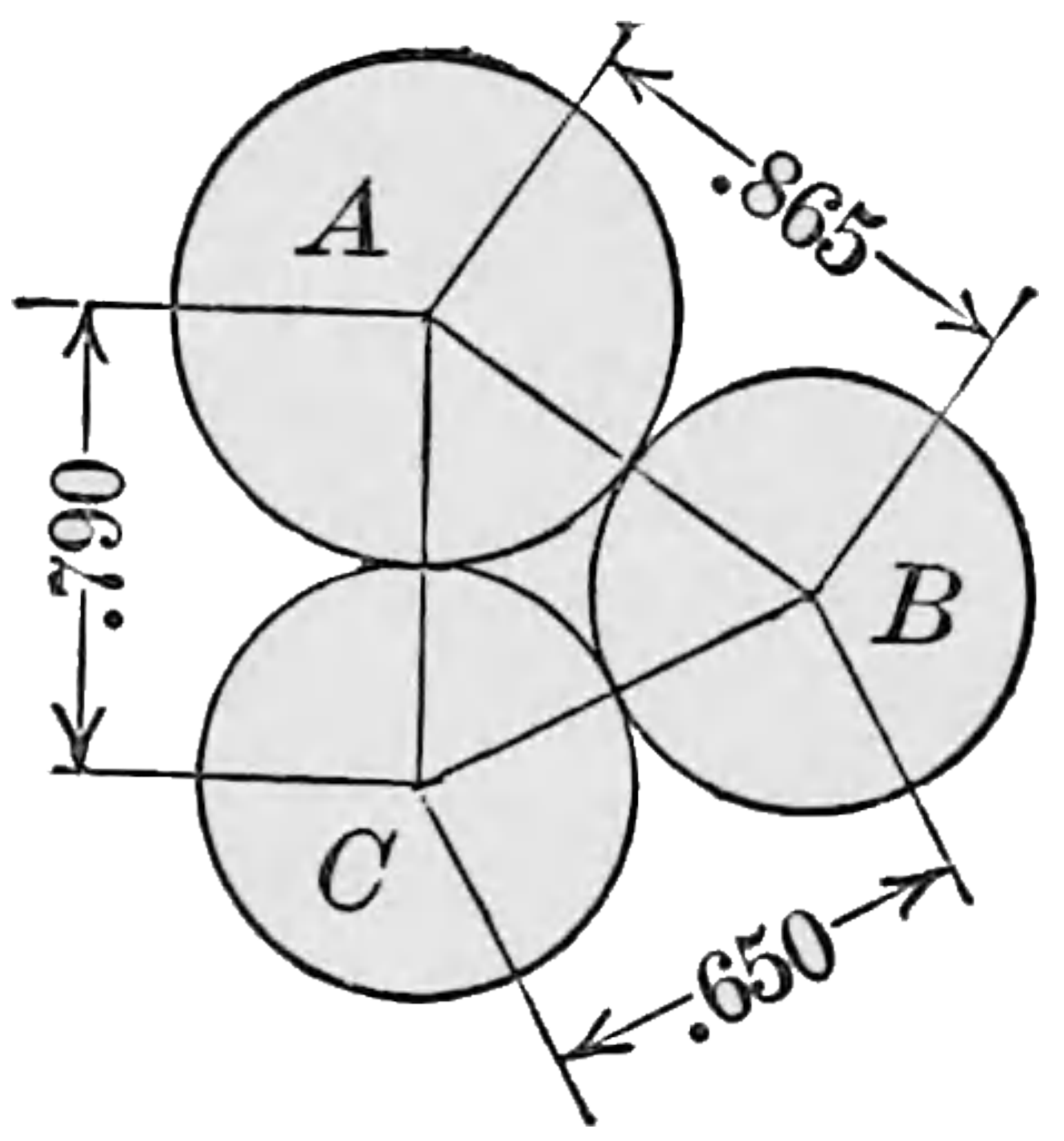
19. The freight charges between two cities on 400 lb. of first-class freight and 600 lb. of second-class freight were $14.24, while the charges on 500 lb. of first-class freight and 800 lb. of second-class were $18.48. What was the rate per 100 lb. on each class?

20. The freight charges on shipments between two places were as follows: 800 lb. of 4th class + 500 lb. of 5th class + 700 lb. of 6th class, $17.11; 1000 lb. of 4th class + 600 lb. of 5th class + 800 lb. of 6th class, $20.66; 600 lb. of 4th class + 1000 lb. of 5th class + 900 lb. of 6th class, $20.52. Find the rate per 100 lb. for each of the classes named.

21. The corn and wheat crops of the United States in the year 1909 were together 3,509,000,000 bu.; the corn and oat crops 3,779,000,000 bu.; and the wheat and oat crops, 1,744,000,000 bu. How many bushels were in each crop?

22. One cubic foot of iron and one cubic foot of aluminum weigh 636 lb.; a cubic foot of iron and one of copper weigh 1030 lb.; a cubic foot of copper and one of aluminum weigh 706 lb. How much does one cubic foot of each of these materials weigh?

23. In boring holes in a metal plate, three circles touching each other are to be drawn, the distances between their centers being .865 in., .650 in., and .790 in., respectively. Find the radius of each of the three circles.



24. The Eiffel Tower is taller than the Metropolitan Life Building of New York, and the latter building is taller than the Washington Monument. If the difference between the heights of the first two is 284 ft.; between the first and last is 429 ft.; and between the last two is 145 ft., find the height of each.

25. A ton of fertilizer which contains 60 lb. of nitrogen, 100 lb. of potash, and 150 lb. of phosphate is worth $21.50; a ton containing 70, 80, and 90 lb. of these constituents in order is worth $19; and one containing 80, 120, 150 lb. of each in order is worth $25.50; what is the value of one pound of each of the constituents named?

26. If a bushel of oats is worth 40¢ and a bushel of corn is worth 55¢, how many bushels of each must a miller use to produce a mixture of 100 bu. worth 48¢ a bushel?

27. How many pounds of 20¢ coffee and how many pounds of 32¢ coffee must be mixed together to make 60 lb. worth 28¢ a pound?

28. Make up and work an example similar to Ex. 27.

29. If two grades of tea worth 50¢ and 75¢ a pound are to be mixed together to make 100 lb. which can be sold for 72¢ at a profit of 20%, how many pounds of each must be used?

30. A farmer wishes to combine milk containing 5% of butter fat with cream containing 40% of butter fat in order to produce 20 gal. of cream which shall contain 25% of butter fat. How many gallons of milk and how many of cream must he use?

31. A man has $5050 invested, part at 4%, and the rest at 5%. How much has he invested at each rate if his annual income is $220?

Can you work this example by use of one unknown quantity?

32. A man wishes to invest part of $12,000 at 5% and the rest at 4% so that he may obtain an income of $500. How much must he invest at each of the rates named?

33. Make up and work an example similar to Ex. 32.

34. If a rectangle were 3 in. longer and 1 in. narrower it would contain 5 sq. in. more than it does now; but if it were 2 in. shorter and 2 in. wider its area would remain unchanged. What are its dimensions?

SUG. Draw a diagram for each rectangle considered in the problem. See Ex. 30, p. 104.

35. If a rectangle were made 3 ft. shorter and $1\frac{1}{2}$ ft. wider, or if it were 7 ft. shorter and $5\frac{1}{4}$ ft. wider, its area would remain unchanged. What are its dimensions?

36. A party of boys purchased a boat and upon payment for the same discovered that if they had numbered 3 more, they would have paid a dollar apiece less; but if they had numbered 2 less, they would have paid a dollar apiece more. How many boys were there, and what did the boat cost?

SUG. Let x = the number of boys, and y = the number of dollars each paid. Then xy represents the number of dollars the boat cost.

37. After going a certain distance in an automobile, a driver found that if he had gone 3 mi. an hour faster, he would have traveled the distance in 1 hr. less time; and that if he had gone 5 mi. faster, he would have gone the distance in $1\frac{1}{2}$ hr. less. What was the distance?

38. Make up and work an example similar to Ex. 37.

39. If a baseball nine should play two games more and win both, it will have won $\frac{2}{3}$ of the games played. If, however, it should play 5 more and win 4 of them, it will also have won $\frac{2}{3}$ of the games played. How many games has it so far played and how many has it won?

40. If a physician should have 12 more cases of diphtheria and treat 10 of them successfully, he will have treated $\frac{3}{4}$ of his cases successfully, But if he should have 32 more cases and succeed with 30 of them, he will have succeeded with $\frac{4}{5}$ of his cases. How many cases has he had so far and how many has he treated successfully.

41. If 1 be added to the numerator of a certain fraction, the value of the fraction becomes $\frac{1}{3}$; but if 1 be subtracted from its denominator, the value of the fraction becomes $\frac{1}{4}$. Find the fraction.

42. There is a fraction such that if 4 be added to its numerator the fraction will equal $\frac{4}{5}$; but if 3 be subtracted from its denominator the fraction will equal $\frac{2}{3}$. What is the fraction?

43. Make up and work an example similar to Ex. 42.

44. A certain fraction becomes equal to $\frac{4}{9}$ if $1\frac{3}{5}$ is added to both numerator and denominator. It becomes $\frac{1}{5}$ if $2\frac{1}{4}$ is subtracted from both numerator and denominator. What is the fraction?

45. Find two fractions, with numerators 11 and 7, respectively, such that their sum is $3\frac{1}{12}$, but when their denominators are interchanged, their sum becomes $3\frac{7}{12}$.

46. If $\frac{2}{3}$ is added to the numerator of a certain fraction, its value is increased by $\frac{2}{21}$; but if $2\frac{1}{3}$ is taken from its denominator, the fraction becomes $\frac{6}{7}$. Find the fraction.

47. The sum of two numbers is 97, and if the greater is divided by the less, the quotient is 5 and the remainder 1. Find the numbers.

Sug. The divisor multiplied by the quotient is equal to the dividend diminished by the remainder.

48. Divide the number 100 into two such parts that the greater part will contain the less 3 times with a remainder of 16.

49. The difference between two numbers is 40, and the less is contained in the greater 3 times with a remainder of 12. Find the numbers.

50. Separate 50 into two such parts that $\frac{1}{4}$ of the larger shall exceed $\frac{1}{3}$ of the smaller by 2.

51. A tank can be filled by two pipes one of which runs 4 hr. and the other 5; or by the same two pipes if the first runs 3 hr. and the other 8. How long will it take each pipe running separately to fill the tank?

52. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when B is left alone, and completes the task in 36 days. In what time could each do the work separately?

53. A and B can do a piece of work in 8 da.; A and C can do the same in 10 da.; and B and C can do it in 12 da. How long will it take each to do it alone?

54. 37 means $10 \times 3 + 7$. Does xy mean $10x + y$? Why this difference?

55. How would you write a number whose digits in order from left to right are l, m, and n? Why may not such a number be expressed as lmn?

56. Express in symbols a number whose digits in order are a, b, c, and d. Whose digits are x, y, and z. x and y.

57. A number consists of two digits whose sum is 13, and if 4 is subtracted from double the number, the order of the digits is reversed. Find the number.

58. The sum of the digits of a certain number of two figures is 5, and if 3 times the units' digit is added to the number, the order of the digits will be reversed. What is the number?

59. Twice the units' digit of a certain number is 2 greater than the tens' digit; and the number is 4 more than 6 times the sum of its digits. Find the number.

60. In a number of 3 figures, the first and last of which are alike, the tens' digit is one more than twice the sum of the other two, and if the number is divided by the sum of its digits, the quotient is 21 and the remainder 4. Find the number.

61. An oarsman can row 12 mi. down stream in 2 hr., but it takes him 6 hr. to return against the current. What is his rate in still water and what is the rate of the stream?

Make up and work a similar example.

62. A boatman rows 20 mi. down a river and back in 8 hr.; he can row 5 mi. down the river while he rows 3 mi. up the river. Find the rate of the man and of the stream.

63. A man rows down a stream 20 mi. in $2\frac{2}{3}$ hr., and rows back only $\frac{3}{5}$ as fast. Find the rate of the man and of the stream.

64. 3 cu. ft. of cast iron and 5 cu. ft. of wrought iron together weigh 3750 lb.; also 7 cu. ft. of the former and 4 cu. ft. of the latter weigh 5070 lb. What is the weight of 1 cu. ft. of each?

65. Regarding the orbits of the earth and of the planet Mars as circles whose center is the sun, the greatest distance between the earth and Mars at any time is 234,000,000 mi., and the least distance between them is 48,000,000 mi. How far is each of them from the sun?

66. 2 lb. of tea and 5 lb. of coffee cost \$2.50. If the price of tea should increase 10% and that of coffee should diminish 10%, the cost of the above amounts of each would be \$2.45. Find the cost of a pound of each.

67. Two bins contain a mixture of corn and oats, the one twice as much corn as oats, and the other three times as much oats as corn. How much must be taken from each bin to fill a third bin holding 40 bu., to be half oats and half corn?

68. If A gives B \$10, A will have half as much as B; but if B gives A \$30, B will have $\frac{4}{5}$ as much as A. How much has each?

69. Two grades of spices worth 25¢ and 50¢ a pound are to be mixed together to make 200 lb. which can be sold at 52¢ per lb. at a profit of 30%. How many pounds of each grade must be used?

70. A train maintained a uniform rate for a certain distance. If this rate had been 8 mi. more each hour, the time occupied would have been 2 hr. less; but if the rate had been 10 mi. an hour less, the time would have been 4 hr. more. Find the distance.

71. If the greater of two numbers is divided by the less, the quotient is 3 and the remainder 3, but if 3 times the greater be divided by 4 times the less, the quotient is 2 and the remainder 20. Find the numbers.

72. Why are we able to solve problems like Exs. 70 and 71 by algebra and not by arithmetic?

73. Find two numbers whose sum is a and whose difference is b.

74. If a pounds of sugar and b pounds of coffee together cost c cents, while d pounds of sugar and e pounds of coffee together cost f cents, what is the price of one pound of each?

75. If a bushel of oats is worth p cents, and a bushel of corn is worth q cents, how many bushels of each must be mixed to make r bushels worth s cents per bushel?

76. Find a fraction such that if a be added to both numerator and denominator the value of the fraction is p/q; but if b is added to both numerator and denominator, the value of the fraction is r/s.

77. Generalize Ex. 34 (p. 242), by using a letter for each number in the example.

78. Generalize Ex. 53 (p. 245), by using a letter for each number in the example.

79. Make up and work three examples similar to such of the examples in this Exercise as you think are most interesting or instructive.

143. Utilities in Algebra.

1. *Brevity of expressions which represent numbers.* Brevity means a *saving of time and energy.*

Thus, for instance, for " number of feet in the length of a rectangle," we may use a single letter as x.

2. *The saving of space* also opens the way for the *use of auxiliary quantity of* various kinds.

See, for instance, the process of factoring $a^4 + a^2b^2 + b^4$, p. 153.

3. By using a letter to represent any number (of a given class), we are able to *discover and prove general laws of numbers.*

Thus, $(a + b)^2 = a^2 + 2ab + b^2$ is true for any numbers whatever.

As an example of the discovery of new and useful laws of number, we may take the case where we know half the sum and half the difference of two numbers and desire to find the numbers themselves. In the above description of the known facts, there is nothing to suggest a method of obtaining the desired end. But if we express the given facts in the algebraic form, thus, $\frac{a+b}{2}$ and $\frac{a-b}{2}$, it is at once suggested that half the difference *added* to half the sum will give a, the greater of the two numbers, and *subtracted* will give b, the smaller.

It may be well to notice that one source of this discovery is that in the algebraic expression we used separate symbols, a and b, of nearly equal size for the two numbers considered.

4. *Combination of several rules into one formula.*

Thus, the single formula $p = br$ combines three cases (and rules) used in arithmetic in treating percentage. Similarly, the formula $i = prt$ covers all cases used in treating interest in arithmetic.

This advantage comes (1) from the fact that a letter may be used to represent any number. See 3 above.

(2) From the fact that an equation can be solved for any letter in the equation.

(3) From the approximately uniform size of the letters employed, which suggests that we treat all the letters alike and give each the leadership in turn.

5. The use of letters to represent unknown numbers often enables us to *begin in the middle of a complex problem and work*

in several directions and thus solve problems which otherwise we could not analyze. See the examples on pp. 242–243.

6. We should also remember constantly that the symbols used in algebra (and the advantages coming from their use) are but a part, or detail, of the more general subject of symbolism as a whole and of its utilities; and that a *training in algebra should give a better grasp of the whole subject of symbols and their uses.*

EXERCISE 76

1. Abbreviate the following as much as you can by use of the letter x:

$\frac{1}{2}$ a certain number + $\frac{1}{3}$ the number = 25. How much shorter is your expression than the given expression?

2. Make up and work an example similar to Ex. 1.

3. Why does a knowledge of algebra suggest to us that a number like 27001 can be factored and also the method of doing this, while a knowledge of arithmetic does not do the same? (See Ex. 29, p. 127.)

4. Is a railroad ticket a symbol or representative of the money paid for it? What are the advantages in the use of the ticket? The disadvantages?

5. Discuss in the same way a check drawn on a bank and used in paying a bill.

6. In canceling a railroad ticket, what are the advantages in punching the ticket as compared with crossing it with a pencil mark? With burning it?

7. What is a newspaper (or a book) a symbol or representative of? What are the advantages in its use? The disadvantages?

8. A certain firm occupied a building running from 10 to 20 Barclay St. in a certain city as their place of business. In advertising in one magazine they gave their address as 10 Barclay St.; in another they gave their address as 12 Barclay St.; in another as 14 Barclay St. What was the advantage in doing this? By this means what double use was made of the symbols 10, 12, 14, etc.

9. If a teacher has a set of papers from each of several classes, what is the advantage in arranging them at different angles when piling one set upon another?

10. Can you give another instance where difference of position is utilized as a symbol?

11. What are the advantages in using a flag as a symbol or representative of a nation?

12. What are the advantages and disadvantages of reading a book of travels as compared with traveling?

13. Why does a policeman in a large city have a number as well as a name? Name other classes of men which have numbers as well as names.

14. What are the advantages to a person in having a name?

15. Let each pupil make up (or collect) and work as many examples as possible similar to the examples in this Exercise.

SUG. This work is of such a nature that it may readily be extended in various directions at the option of the teacher.

CHAPTER XIII

GRAPHS

144. A variable is a quantity which has an indefinite number of different values.

A **function** is a variable which depends on another variable for its value.

Thus, the *area* of a circle is a function of the *radius* of the circle; the *wages* which a laborer receives is a function of the *time* that the man works.

A **graph** is a diagram representing the relation between a function and the variable on which the function depends for its value.

A function may depend for its value on more than one variable; thus, the area of a rectangle depends on two quantities — the length of the rectangle and the breadth. The present treatment of graphs, however, is limited to functions which depend on a single variable.

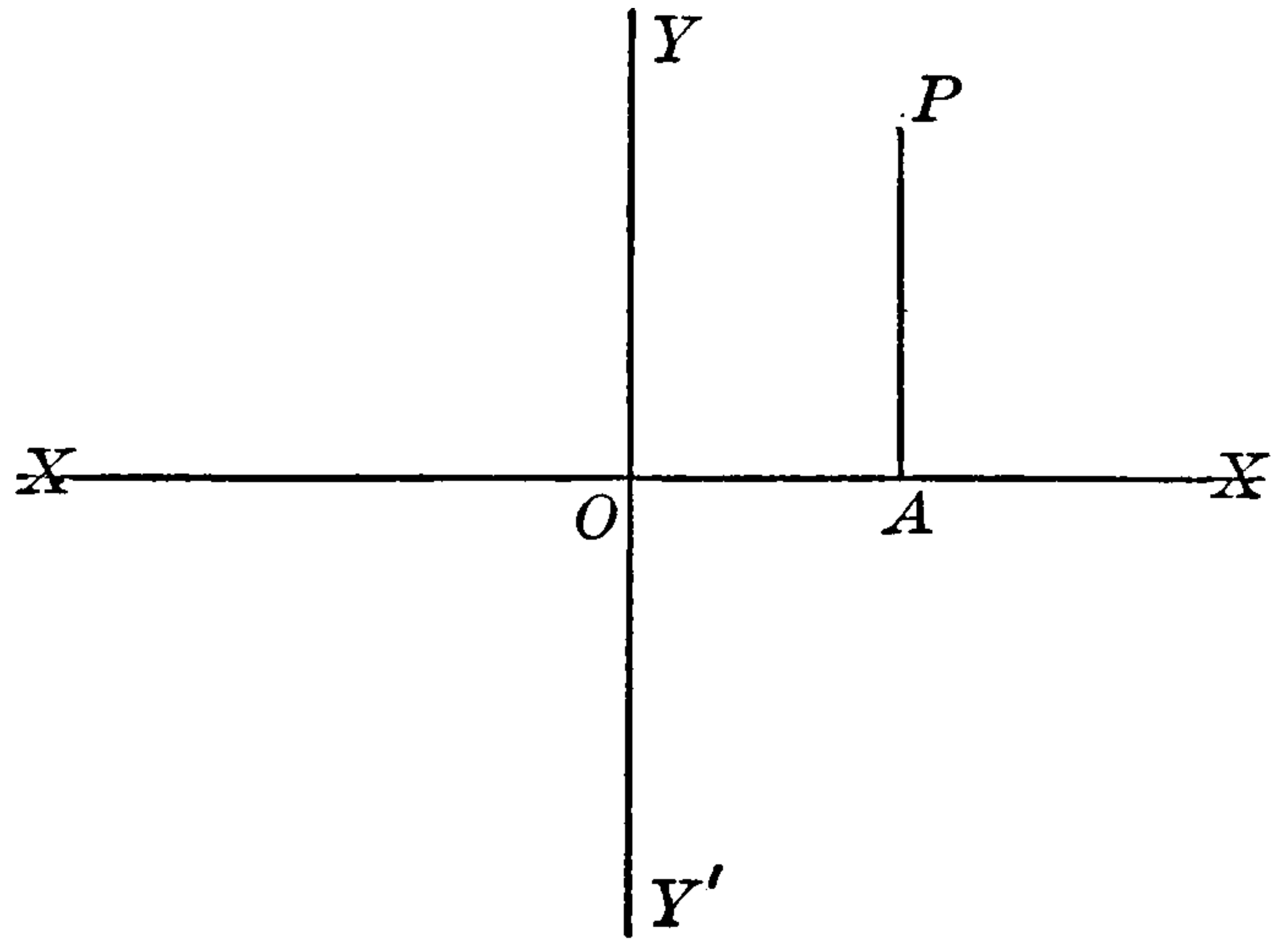
In algebra we study only those functions which have a definite value for each definite value of the variable.

145. Uses of Graphs. A graph is useful in showing at a glance the place where the function represented has the greatest or least value and where it is changing its value most rapidly, and in making clear similar properties of the function.

Graphs of algebraic equations are useful in making clear certain properties of such equations which are otherwise difficult to understand. A graph also often furnishes a rapid method of determining the root (or roots) of an equation.

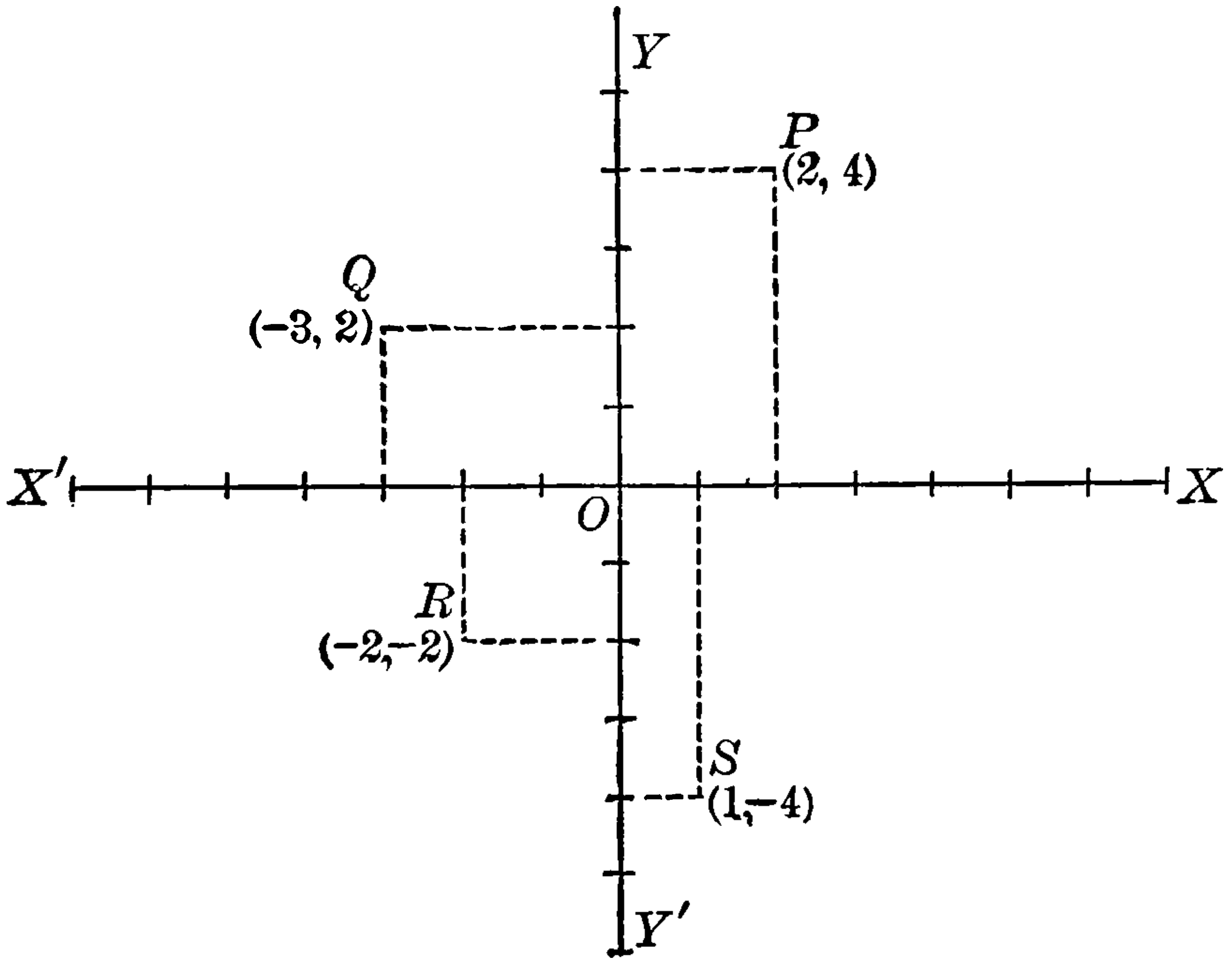
146. Framework of Reference. **Axes** are two straight lines perpendicular to each other which are used as an auxiliary framework in constructing graphs; as XX' and YY'.

The **x-axis, or axis of abscissas,** is the horizontal axis; as XX'. The **y-axis, or axis of ordinates,** is the vertical axis; as YY'.



The **origin** is the point in which the axes intersect; as the point O.

The **ordinate of a point** is the line drawn from the point parallel to the y-axis and terminated by the x-axis. The **abscissa of a point** is the part of the x-axis intercepted between the origin and the foot of the ordinate. Thus, the ordinate of the point P is AP, and the abscissa is OA.



The ordinate is sometimes termed the "y" of a point, and the abscissa, the "x" of a point.

Ordinates above the x-axis are taken as plus; those below, as minus. Abscissas to the right of the origin are plus; those to the left are minus.

The **co-ordinates of a point** are the abscissa and the ordinate taken together. They are usually written together in parenthesis with the abscissa first and a comma between.

Thus, the point (2, 4) is the point whose abscissa is 2 and ordinate 4, or the point **P** of the figure. Similarly, the point $(-3, 2)$ is Q; $(-2, -2)$ is R; and $(1, -4)$ is S.

The **quadrants** are the four parts into which the axes divide a plane. Thus, the points P, Q, R, and S lie in the *first*, *second*, *third* and *fourth* quadrants, respectively.

EXERCISE 77

Draw axes and locate each of the following points:

1. $(3, 2)$, $(-1, 3)$, $(-2, -4)$, $(4, -1)$.
2. $(2, \frac{1}{2})$, $(-3, -1\frac{1}{2})$, $(5, -\frac{5}{4})$, $(-2, \frac{1}{3})$.
3. $(2, 0)$, $(-3, 0)$, $(0, 4)$, $(0, -\frac{1}{2})$, $(0, 0)$.
4. $(1, \sqrt{2})$, $(1, -\sqrt{2})$ $(\sqrt{3}, 0)$ $(\sqrt{5}, -3)$, $(-\frac{1}{2}\sqrt{5}, 2\sqrt{2})$.
5. Construct the triangle whose vertices are $(1, 1)$, $(2, -2)$ $(3, 2)$.
6. Construct the quadrilateral whose vertices are $(2, -1)$, $(-4, -3)$, $(-3, 5)$, $(3, 4)$.
7. Plot the points $(0, 0)$, $(1, 0)$, $(2, 0)$, $(5, 0)$, $(-1, 0)$, $(-3, 0)$.
8. Also $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 5)$, $(0, -1)$, $(0, -3)$.
9. All points on the x-axis have what ordinate?
10. All points on the y-axis have what abscissa?
11. Plot the following pairs of points and find the distance between each pair of points:

(*1*) $(6, 5)$, $(2, 8)$ (*3*) $(3, -6)$, $(-2, 6)$
(*2*) $(3, 0)$, $(0, 6)$ (*4*) $(0, 0)$, $(-3, 5)$

12. Construct the rectangle whose vertices are $(1, 3)$, $(6, 3)$, $(1, -2)$, $(6, -2)$, and find its area.

13. Construct the rectangle whose vertices are $(-3, 4)$, $(4, 4)$, $(-3, -2)$, $(4, -2)$, and find its area.

14. Construct the triangle whose vertices are $(-3, -4)$, $(-1, 3)$, $(2, -4)$, and find its area.

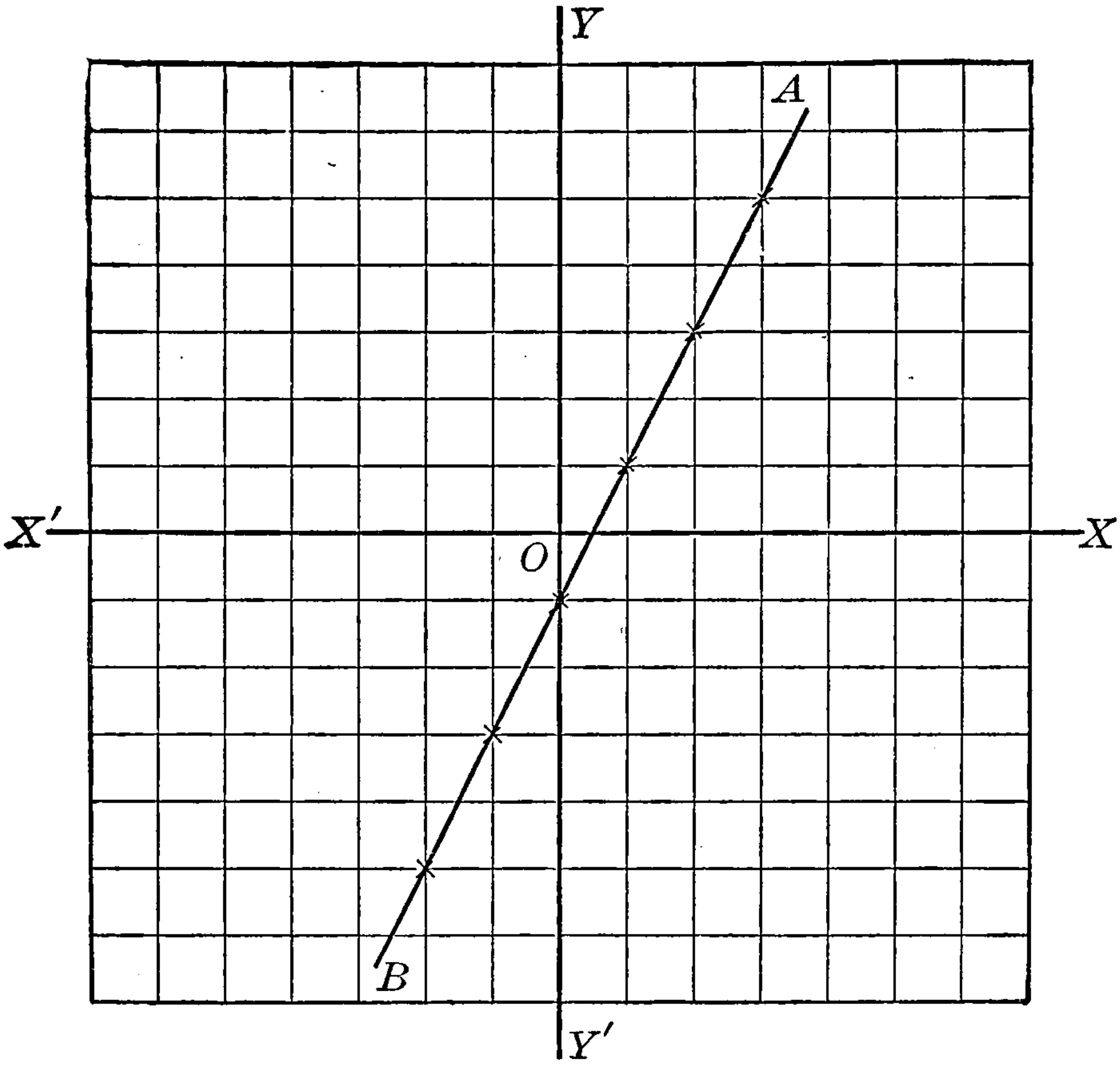
15. In which quadrant are the abscissa and ordinate both plus? Both minus? In which quadrant is the abscissa minus and the ordinate plus? In which is the abscissa plus and the ordinate minus?

16. Practice oral work with small fractions as in Exercise 58 (p. 190).

GRAPHS OF EQUATIONS OF THE FIRST DEGREE

147. To Construct the Graph of an Equation of the First Degree Containing Two Unknown Quantities, as x and y,

Let x have a series of convenient values, as 0, 1, 2, 3, etc., -1, -2, -3, etc.;



Find the corresponding values of y;

Locate the points thus determined, and draw a line through these points.

Ex. Construct the graph of the equation $y = 2x - 1$.

Construct the points $(0, -1)$, $(1, 1)$, $(2, 3)$, $(3, 5)$, $(-1, -3)$, $(-2, -5)$, etc., and draw a line through them. The straight line AB is thus found to be the graph of $y = 2x - 1$.

x	y
0	−1
1	1
2	3
3	5
etc.	etc.
−1	−3
−2	−5
etc.	etc.

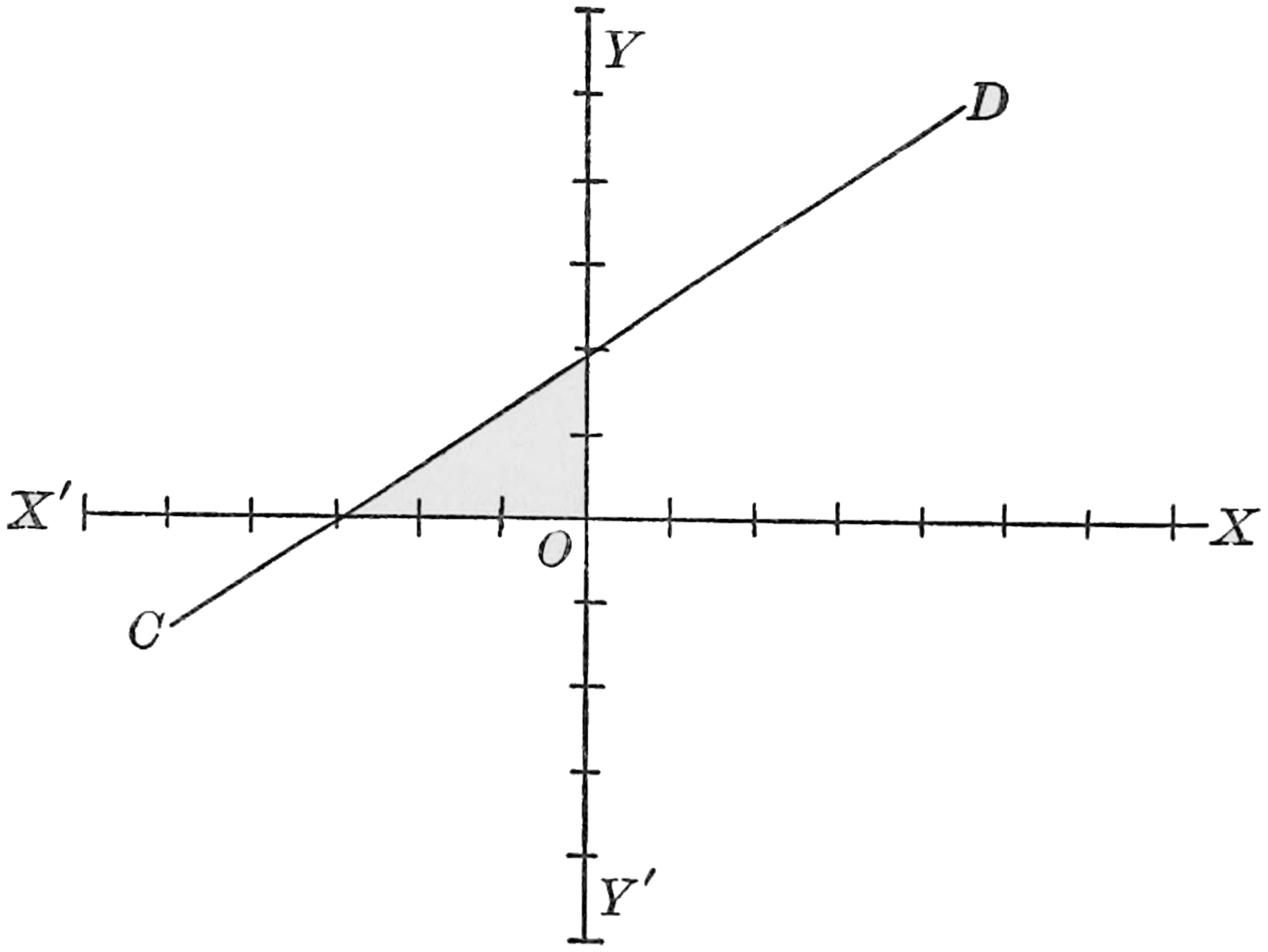
148. Linear Equations. It will always be found that the graph of an equation of the first degree containing not more than two unknown quantities is a straight line. Hence,

A **linear equation** is an equation of the first degree.

149. Abbreviated Method of Constructing the Graph of a Linear Equation. Since a straight line is determined by two points, in order to construct the graph of an equation of the first degree it is sufficient to *construct any two points of the graph and draw a straight line through them.*

Ex. 1. Graph $3y - 2x = 6$.

When $x = 0$, $y = 2$; when $y = 0$, $x = -3$. Hence, the graph passes through the points $(0, 2)$ and $(-3, 0)$, or CD is the required graph.



The greater the distance between the points chosen, the more accurate the construction will be. It is usually advisable to test the result obtained by locating a third point and observing whether it falls upon the graph as constructed.

If the given line does not pass through the origin, or near the origin on both axes, it is often convenient to construct the line by determining the points where the line crosses the axes.

Ex. 2. Graph $4x + 7y = 1$.

When $x = 0, y = \frac{1}{7}$; when $y = 0$, $x = \frac{1}{4}$. Hence, the graph passes close to the origin on both axes. Hence, find two points on the required graph at some distance from each other, as by letting $x = 0, 9$, and finding $y = \frac{1}{7}, -5$. Let the pupil construct the figure.

EXERCISE 78

Graph the following. (It is an advantage, if possible, to draw the graph line in red, the rest of the figure in black ink.)

1. $y = x + 2$

2. $y = x - 2$

3. $3x + 2y = 6$

4. $3x - 2y = 6$

5. $3x - 5y + 15 = 0$

6. $y = 2x$

7. $4x - 5y = 1$

8. $\dfrac{x-1}{2} = 3y$

9. $x = 3(y - 1)$

10. $y = -x$

11. $y = 4$

12. If $x = 2$, show that whatever value y has, x always $= 2$. Hence the graph of $x = 2$ is a line parallel to the y-axis.

13. Graph $x = 0$; also $y = 0$.

14. Show how to determine from an inspection of a linear equation whether its graph passes through the origin; near the origin on one axis; near the origin on both axes.

15. Graph $5x + 6y = 1$; also $6x - y = 12$.

16. Obtain and state a short method of graphing a linear equation in which the term which does not contain x or y is missing, as $2y - 3x = 0$.

Before graphing the following, determine the best method of constructing each graph, and then graph:

17. $x + 2y = 4$

18. $2y = x$

19. $5x - 6y = 1$

20. $\frac{1}{7}x + \frac{1}{2}y = \frac{1}{2}$

21. $x = -3$

22. $5x + 4y = 0$

23. $x - y = 5$

24. $y + 2 = 0$

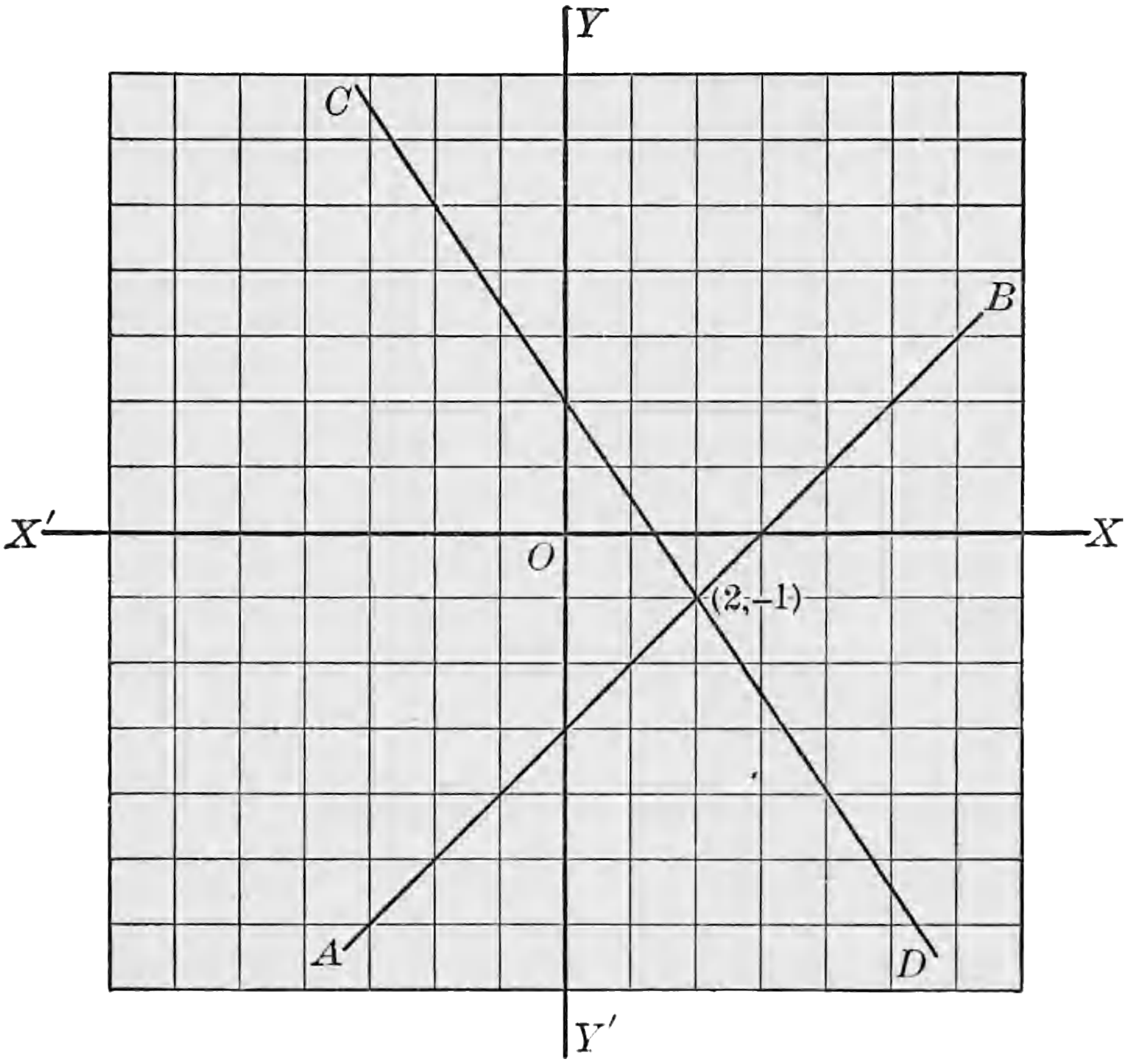
25. $3x - 2y + \frac{11}{2} = 0$

26. Construct the triangle whose sides are the graphs of the equations, $y - 2x + 1 = 0$, $3y - x - 7 = 0$, $y + 3x + 11 = 0$.

27. An equation of the form $y = b$ represents a line in what position? One of the form $x = a$?

28. Make up and work an example similar to Ex. 4. To Ex. 29.

150. Graphic Solution of Simultaneous Linear Equations. If we construct the graph of the equation $x - y = 3$ (the line AB) and the graph of $3x + 2y = 4$ (the line CD), and



measure the co-ordinates of their points of intersection, we find this point to be (2, -1).

If we solve the pair of simultaneous equations $\begin{cases} x - y = 3 \\ 3x + 2y = 4 \end{cases}$ by the ordinary algebraic method, we find that $x = 2$ and $y = -1$.

In general, *the roots of two simultaneous linear equations correspond to the co-ordinates of the point of intersection of their graphs;* for these co-ordinates are the only ones which satisfy both graphs, and their values are also the only values of x and y which satisfy both equations.

Hence, to obtain the graphic solution of two simultaneous equations,

Draw the graphs of the given equations, and measure the co-ordinates of the point (*or points*) *of intersection.*

Graphing two simultaneous equations forms a convenient method of testing or checking their algebraic solution.

151. Simultaneous Linear Equations whose Graphs are Parallel Lines. Construct the graph of $x + 2y = 2$ and also of $3x + 6y = 12$.

You will find that the graphs obtained are parallel straight lines. Now try to solve the same equations algebraically. You will find that when either x or y is eliminated, the other unknown quantity is eliminated also, and that it is therefore impossible to obtain a solution. The reason why an algebraic solution is impossible is made clear by the fact that the graphs, being parallel lines, cannot intersect; that is to say, there are no values of x and y which will satisfy both of these lines, or both equations, at the same time.

152. Graphic Solution of an Equation of the First Degree of One Unknown Quantity. By substituting for y in the first equation of the pair $\begin{cases} y = x - 3 \\ y = 0 \end{cases}$ the two equations reduce to $x - 3 = 0$. Accordingly, the graphic solution of an equation like $x - 3 = 0$ can be obtained by combining the graphs of $y = x - 3$ and $y = 0$. In other words, the root of $x - 3 = 0$ is represented graphically by the abscissa of the point where the graph of $y = x - 3$ crosses the x-axis.

EXERCISE 79

Solve each pair of the following equations both graphically and algebraically, and compare the results in each example:

1. $\begin{cases} 2x + 3y = 7 \\ x - y = 1 \end{cases}$

2. $\begin{cases} y = 3x - 4 \\ y = -2x + 1 \end{cases}$

3. $\begin{cases} 2y = x \\ x + y + 6 = 0 \end{cases}$

4. $\begin{cases} y = 2x \\ x + y = 0 \end{cases}$

5. $\begin{cases} x + 7y + 11 = 0 \\ x - 3y + 1 = 0 \end{cases}$

6. $\begin{cases} y = 3 \\ 9x - 5y = 3 \end{cases}$

7. $\begin{cases} y = 0 \\ y = 2x + 3 \end{cases}$

8. Solve graphically $2x+3=0$

9. Solve graphically $3x-5=0$

10. Discover and state the relation between the coefficients of two linear simultaneous equations whose graphs are parallel lines.

11. Solve graphically $\begin{cases} 8x + 5y = 7. \\ 6x + 2y = 11. \end{cases}$

12. Solve both algebraically and graphically $\begin{cases} 2x - 3y = 5. \\ 6x - 9y = 5. \end{cases}$

13. Construct the quadrilateral whose sides are the graphs of the equations, $x - 2y - 4 = 0$, $x + y = 1$, $3y - 5x - 15 = 0$, $x + 2y - 4 = 0$, and find the coordinates of the vertices of the quadrilateral.

14. Make up and work an example similar to Ex. 1. To Ex. 6.

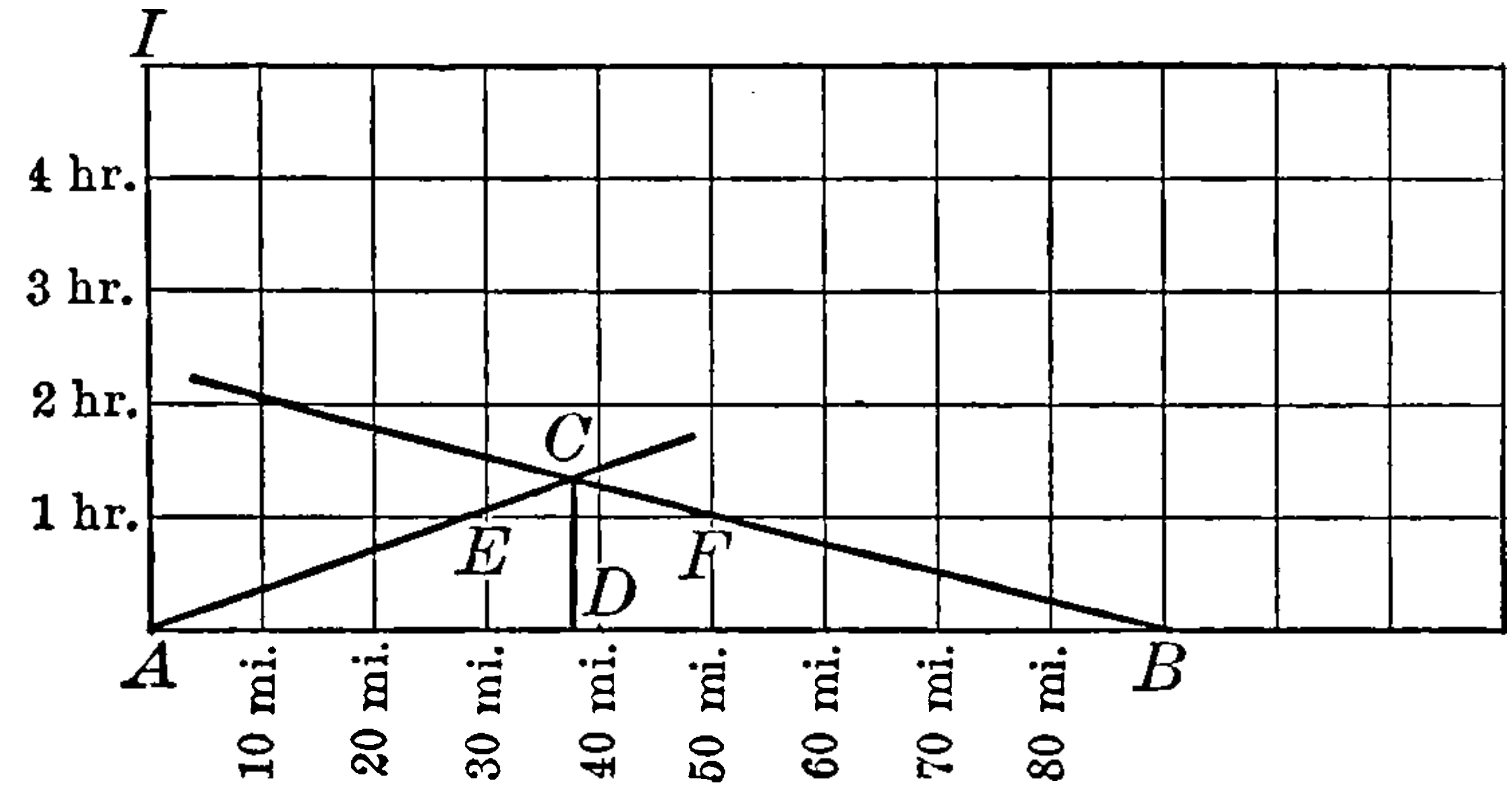
15. How many examples in Exercise 26 (p. 110) can you now work at sight?

153. Graphic Solution of Written Problems.

I. Railway Problems.

Ex. The distance between New York and Philadelphia is 90 mi. At a given time, a train leaves each city, bound for the other city, the train from New York going at 40 mi. an hour and the one from Philadelphia at 30 mi. In how many hours will they meet, and at what distance from New York?

The train dispatcher represents the distance between the stations by the line *AB*, each space denoting 10 mi. Each space on *AI* represents 1 hour. He locates *E* three units to the right of *A* and one unit above *AB*, and *F* four units to the left of *B* and one unit above *AB*. He produces *AE* and *BF* to meet at *C*, and draws *CD* perpendicular to *AB*.



He obtains the distance from *A* at which the trains meet, by measuring *AD* to scale (and hence determines the siding at which one train must wait for the other). He obtains the time that elapses before the trains meet, by measuring *CD* to scale.

The problem may also be solved algebraically in the same way as Exs. 57–61, p. 87.

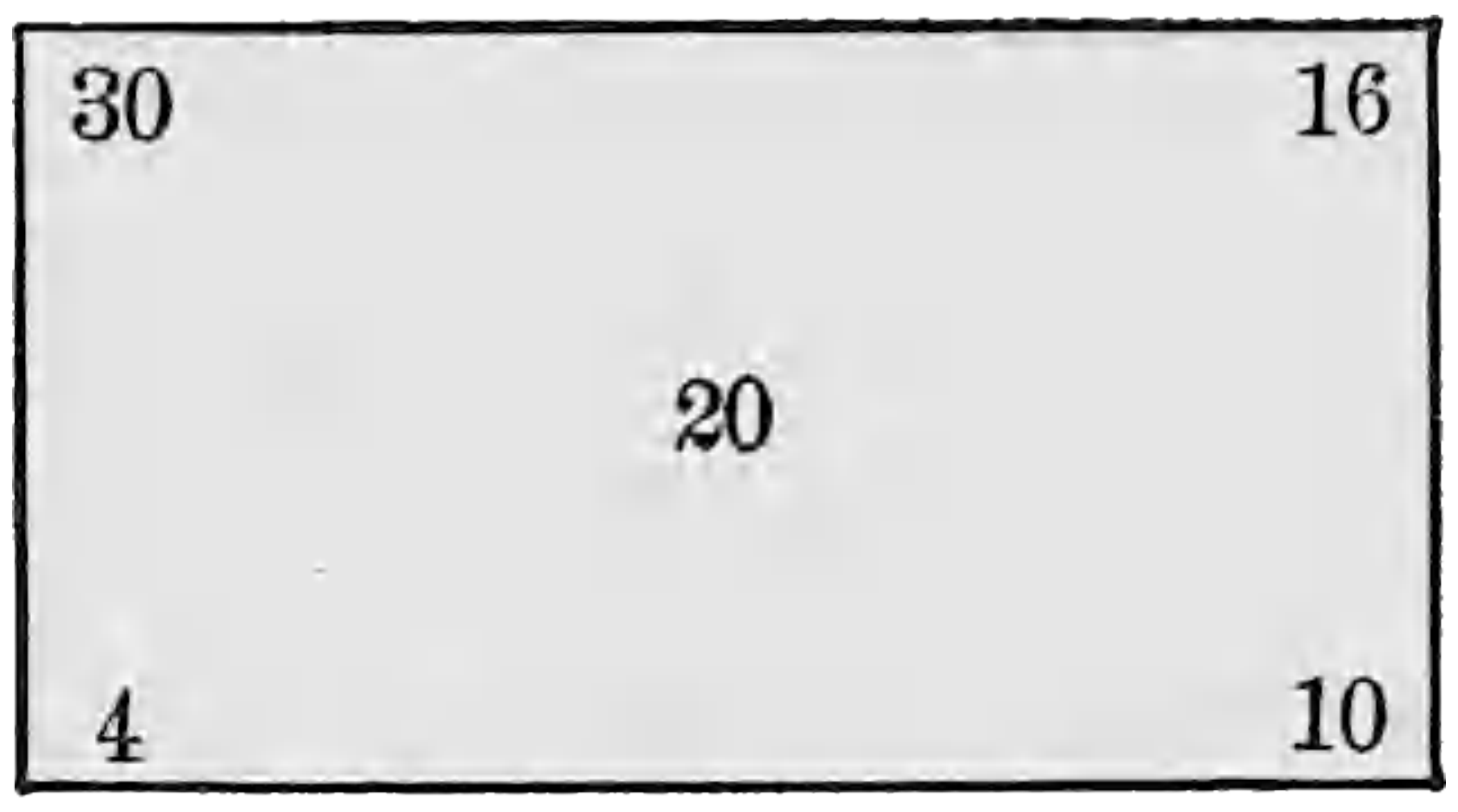
The advantage of the graphical method is that in this solution it is easy to make allowance for any waits which trains may make at stations. Hence, railroad time-tables are often constructed entirely by graphical methods.

II. Problems in the Mixture of Materials.

Ex. In order to obtain a mixture containing 20% of butter fat, in what proportion should cream containing 30% of fat be mixed with milk containing 4%?

Graphical Solution

We construct a rectangle, and write in two adjacent corners (here the left-hand corners) the per cents of fat (30 and 4) in the two given fluids; and in the middle of the rectangle we write the per cent (20) desired in the mixture. The differences between the number in the middle and the numbers in the corners (16 and 10) are then found and placed as in the diagram. The differences thus found show the relative amounts of the given fluids to be used, viz.: 10 parts of cream, and 16 of milk.



Now solve this problem algebraically by the method used for Exs. 26–30, pp. 241–242; and by an examination of this solution, discover for yourself the reason for the above graphical solution.

EXERCISE 80

Solve the following problems graphically:

1. The distance between New York and Philadelphia is 90 mi. If a train leaves New York at noon and goes 40 mi. an hour, and another train leaves Philadelphia at the same time and travels 20 mi. an hour, at what time and how far from New York will they meet?

2. Make up and work an example similar to Ex. 1.

3. The distance between New York and Buffalo is 440 mi. If a train leaves New York at 11 A. M. and travels at the rate of 40 mi. an hour, and a train traveling 30 mi. an hour leaves Buffalo at the same time, at what time and how far from New York will the trains meet?

4. Make up and work an example similar to Ex. 3.

5. In order to obtain a mixture containing 22% butter fat, in what proportion must cream containing 32% of fat be mixed with milk containing 5%?

6. In order to obtain a mixture containing 28% of butter fat, in what proportion must cream containing 35% of fat be mixed with cream containing 25%?

7. Make up and work an example similar to Ex. 6.

8. In what proportion must oats worth 50¢ a bushel be mixed with corn worth 80¢ a bushel in order to make a mixture worth 60¢ a bushel?

9. Make up and work a similar example concerning mixing different grades of coffee.

10. The distance *PQ* is 48 mi. At 8 A. M. one boy starts from *P* and walks toward *Q* at the uniform rate of 4 mi. an hour. At the same time another boy starts from *Q* on a bicycle and rides toward *P* at the rate of 8 mi. an hour but at the end of each hour of riding rests $\frac{1}{2}$ an hour. By means of a graph determine where and when the two boys will meet.

11. Make up and work an example similar to Ex. 10.

12. How many examples in Exercise 27 (p. 112) can you now work at sight?

EXERCISE 81

Review

1. Tell the degree of each term of

$$5x^3 - 4x^2y^2 - 11x - xy + xy^3 - x^4y + 3x^2 - 7y + 11.$$

2. Factor (1) $x^4 + 4$.

(2) $m^2 - 2mn + n^2 + 5m - 5n$.

(3) $a^2 - n^2 - m^2 - 2ab + 2mn + b^2$.

3. Factor $2(x^3 - 1) + 7(x^2 - 1)$.

Simplify:

4. $$\frac{4x-5}{45} - \frac{4+x}{30} + \frac{2}{3} - \frac{x-5}{18}.$$

5. $$\frac{1-5x}{6x^2-6} + \frac{3x+5}{4x+4} + \frac{2x-3}{3-3x}.$$

6. $$\frac{2}{(x-1)^3}+\frac{1}{(1-x)^2}-\frac{2}{1-x}-\frac{1}{x}.$$

7. $$\frac{\frac{3}{5}(\frac{2}{3}x^2-\frac{11}{6}x-\frac{5}{2})}{\frac{1}{4}(\frac{4}{3}x^2-\frac{2}{3}x-2)}.$$

8. $$\frac{\left(x+\frac{1}{x}\right)^2-2\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2}.$$

9. $$1-\cfrac{x}{a-\cfrac{a}{1-\cfrac{a}{a+x}}}.$$

Solve:

10. $$1\tfrac{5}{6}+x=\frac{x-1}{2}+\frac{1}{3}-\frac{x-3}{5}.$$

11. $$3+\frac{x}{3}-\frac{1}{2}\left(4-\frac{x}{2}\right)=\frac{1}{3}\left\{\frac{x}{3}-\left(2x-\frac{x+1}{2}\right)\right\}.$$

12. $$\frac{3x-2}{6}-\frac{5x+14}{7x+15}-\frac{3-2x}{4}=x-1\tfrac{1}{4}.$$

13. $$\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}.$$

14. $$\frac{3x}{2}-\frac{y}{3}=\frac{4}{9}.$$
$$\frac{5x}{4}+\frac{3y}{2}=3\tfrac{1}{3}.$$

15. $$2y-x=4xy.$$
$$\frac{4}{y}-\frac{3}{x}=9.$$

16. $$\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}.$$
$$\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}.$$

17. $$(a-b)x+(a+b)y=a+b.$$
$$(x-y)(a^2-b^2)=a^2+b^2.$$

18. $$.3x+.2y=1.3.$$
$$.3y+.2z=.8.$$
$$.3z+.2x=.9.$$

19. $$\frac{x}{y}=\frac{5}{6}.$$
$$x-y=-\frac{2}{15}.$$

20. Solve $$\frac{x}{a}+\frac{y}{b}=\frac{1}{ab}.$$
$$\frac{x}{a'}+\frac{y}{b'}=\frac{1}{a'b'}.$$

21. Is it allowable to divide each term in $16x = 96$ by 16? Is it allowable to divide each term of $16x - 96$ by 16? Give reasons.

22. Obtain the value of

$$\frac{ax+1}{x}-\left[a(x+1)-\frac{a(x^2-1)-x}{x}\right], \text{ when } x=\frac{a-1}{a}.$$

23. Solve $\frac{1}{a+b}+\frac{a+b}{x}=\frac{1}{a-b}+\frac{a-b}{x}$.

24. Why is it proper to change $-x=-3$ into $x=3$, and not proper to change $-x-3$ into $x+3$?

25. What number added to the denominators of $\frac{a}{b}$ and $\frac{c}{d}$, respectively, will make the results equal?

26. Does $\frac{c}{a-b}$ equal $\frac{c}{a}-\frac{c}{b}$? Does $\frac{a+b}{c}$ equal $\frac{a}{c}+\frac{b}{c}$? Does $\frac{c}{a+b}$ equal $\frac{c}{a}+\frac{c}{b}$? Verify your statements for the special case when $a=4$, $b=8$, and $c=2$.

27. The sums of three numbers taken two and two are 20, 29, and 27. What are the numbers?

28. Factor $8(x+y)^3-(2x-y)^3$.

29. What is the advantage of being able to add the same number to both members of an equation? In being able to transpose a term? To divide both members of an equation by the same number? (See Art. 70.)

30. Solve

$$(a+c)x-(a-c)y=2ab.$$
$$(a+b)x-(a-b)y=2ac.$$

31. Does $(a^2+b^2)(a+b)$ equal a^3+b^3? Verify your statement by letting a and b have convenient numerical values. Can you prove your statement without the use of numbers?

32. If $K=\pi R^2$ and $C=2\pi R$, find K when $C=10$ and $\pi=\frac{22}{7}$.

33. If $s=\frac{1}{2}gt^2$ and $v=gt$, find s when $g=32$ and $v=64$.

34. If $C=2\pi R$ and $V=\frac{4}{3}\pi R^3$, find V when $C=33$ and $\pi=\frac{22}{7}$ (use cancellation wherever possible).

Solve:

35. $4(x+y)+\frac{1}{x-y}=13.$

$3(x+y)-\frac{10}{x-y}=-1.$

36. $3x+\frac{6}{2x-3y}=6.$

$4x-\frac{9}{2x-3y}=25.$

Sug. Let $p=(x+y)$, $q=\frac{1}{x-y}$.

37. Show that $\dfrac{12t(t^3+2)^2(2t^2-3)^2-6t^2(2t^2-3)^3(t^3+2)}{(t^3+2)^4}$ reduces to $\dfrac{6(3t^2+4t)(2t^2-3)^2}{(t^3+2)^3}$.

38. Graph $y=2x+b$ when $b=1$. On the same diagram graph $y=2x+b$ when $b=2$. When $b=-1$. When $b=0$.

39. Graph $y=ax+2$ when $a=1$. On the same diagram graph $y=ax+2$ when $a=2,\ 3,\ -1,\ -3$.

40. Graph $y=3x+2$, and $y=-\frac{1}{3}x+2$ on the same diagram.

41. Make up and work an example similar to Ex. 38. To Ex. 39.

42. The Fahrenheit reading at the boiling point of alcohol is 95° higher than the Centigrade reading. Find each of the readings.

43. Make up an example similar to Ex. 42, using the fact that ether boils at 96° Fahrenheit.

44. Give the value of $\frac{1}{a}\div a$, $a\div\frac{1}{a}$, $-\frac{2}{5}x^2\div\frac{6}{5}x^2$.

45. What is the reciprocal of $\frac{1}{a}+\frac{1}{b}$?

46. Show that elimination by comparison is a special form of elimination by substitution.

47. Show that elimination by addition and subtraction may also be regarded as a form of elimination by substitution.

48. Eliminate a between the equations $F=Ma$ and $s=\frac{1}{2}at^2$.

49. Given $l=a+(n-l)d$ and $s=\frac{n}{2}(a+l)$, find s in terms of d, n, and l.

SUG. What letter must be eliminated?

50. Eliminate l between $l=ar^{n-1}$ and $s=\dfrac{rl-a}{r-l}$.

CHAPTER XIV

INEQUALITIES

154. The Signs of Inequality are $>$, which is read "is greater than," and $<$, which is read "is less than."

Thus, $a > b$ means that a is greater than b.
$c < b$ means that c is less than b.

Observe that both signs of inequality are written with the opening toward the greater quantity.

155. An Inequality is a statement in symbols that one algebraic expression represents a greater or less number than another; as $x + y < a^2 + b^2$.

Remember that any positive number is greater than any negative number, and that of two negative numbers the smaller is the greater. Thus, $2 > -5$, and $-2 < -3$.

The **first member** of an inequality is the expression on the left of the sign of inequality; the **second member** is the expression on the right of this sign.

156. Inequalities of the Same Kind. Two inequalities are said to be of the *same kind*, or to subsist in the same sense, when the greater member occupies the same relative position in each inequality; that is, is the left-hand member in each, or the right-hand member. Hence, in inequalities of the same kind the signs of inequality point in the same direction.

Thus,

$$\left.\begin{aligned} x &> 2x - 3 \\ \frac{2x+1}{3} &> 5x - 4 \end{aligned}\right\} \text{ are of the same kind;}$$

but

$$\left.\begin{aligned} a &< b \\ 2a &> b - \frac{a}{2} \end{aligned}\right\} \text{ are of opposite kinds.}$$

157. Properties of Inequalities. The following primary properties of inequalities are recognized as true:

(1) **Adding and subtracting quantities.** *An inequality will be unchanged in kind if the same quantity is added to or subtracted from each member.* Hence,

(2) **Terms transposed.** *A term may be transposed from one member of an inequality to the other, provided its sign is changed.*

(3) **Signs changed.** *The signs of all the terms of an inequality may be changed, provided the sign of the inequality is reversed.*

(4) **Positive multiplier.** *An inequality will be unchanged in kind if all its terms are multiplied or divided by the same positive number.*

(5) **Raised to a power.** *An inequality will be unchanged in kind if both members are positive and both are raised to the same power.*

(6) **Equalities combined with inequalities.** *If the members of an inequality are subtracted from equals, the result will be an inequality of the opposite kind. If the members of an inequality are divided into equals, the result will be an inequality of the opposite kind.*

(7) **Inequalities combined.** *If the corresponding members of two inequalities of the same kind are added, or multiplied, the resulting inequality will be of the same kind. But if the members of an inequality are subtracted from, or divided by,*

the corresponding members of another inequality of the same kind, the resulting inequality will not always be of the same kind.

The following are numerical illustrations of the above principles:

(1)
$$\begin{array}{r} 7 > 5 \\ 3 = 3 \\ \hline 10 > 8 \end{array}$$

(2)
$$\begin{array}{r} 12 > 7 - 3 \\ \therefore 12 + 3 > 7 \end{array}$$

(3)
$$\begin{array}{lr} & -7 > -10 \\ & \therefore 10 > 7 \\ \text{or} & 7 < 10 \end{array}$$

(4)
$$\begin{array}{r} 7 > 5 \\ 3 = 3 \\ \hline 21 > 15 \end{array}$$

(5)
$$\begin{array}{lr} & 7 > 5 \\ & \therefore 7^2 > 5^2 \\ \text{or} & 49 > 25 \end{array}$$

(6)
$$\begin{array}{r} 20 = 20 \\ 7 > 5 \\ \hline 13 < 15 \end{array}$$

(7)
$$\begin{array}{lr} & 17 > 15 \\ & 8 > 2 \\ \cline{2-2} \text{Adding,} & 25 < 17 \end{array}$$

$$\begin{array}{lr} & 17 > 15 \\ & 8 > 2 \\ \cline{2-2} \text{Subtracting,} & 9 < 13 \end{array}$$

158. A conditional inequality is an inequality which is true only for certain special values of the letter or letters involved.

Thus, $(3 - x)^2 > (x - 4)^2$ is a conditional inequality, since it may be proved that it is true only when $x > 3\frac{1}{2}$.

An **unconditional or absolute inequality** is an inequality which is true for all possible values of the letter or letters involved.

Thus, $a^2 + b^2 > 2ab$ is an unconditional inequality, since it may be proved to be true for all possible values of a and b (the value zero not being considered in this case).

Hence, the conditional inequality corresponds to the conditional equation, and the unconditional inequality to the equation (see Art. 69, p. 94). As with the equality sign, so with the signs > and <, the particular sense in which each is used is, for the present, to be determined by the context.

159. Solution of Conditional Inequalities.

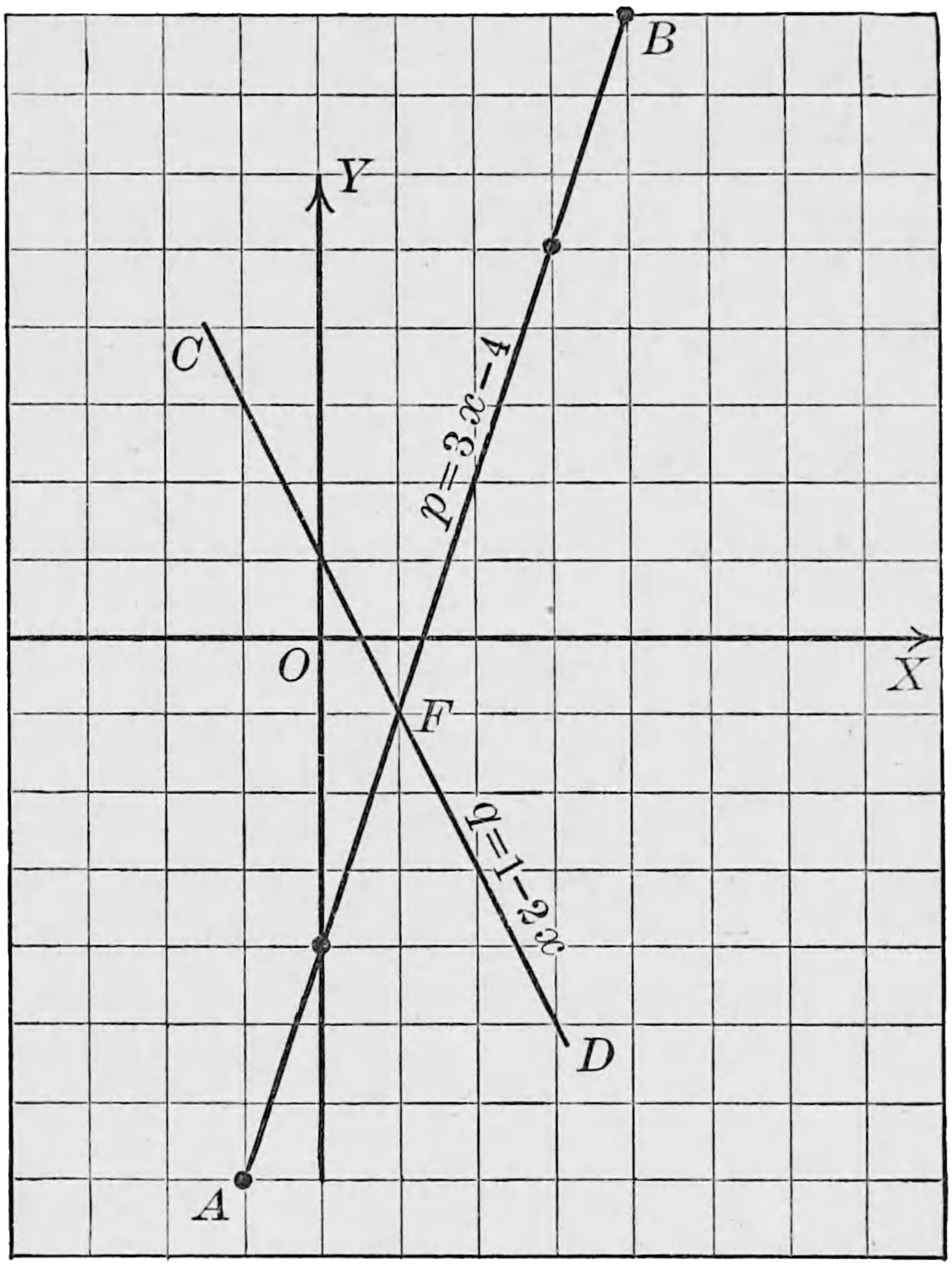
Ex. 1. For what value of x is $3x - 4 > 1 - 2x$?

Transposing terms, $3x + 2x > 5$

$$5x > 5 \qquad \therefore x > 1 \textit{ Ans.}$$

This process may be illustrated graphically as in the diagram.

Graphing $p = 3x - 4$, we obtain line AB. Also graphing $q = 1 - 2x$, we obtain line CD. These lines intersect at the point F, whose abscissa is 1. To the right of F (where $x > 1$), $p > q$, i.e. $3x - 4 > 1 - 2x$.



Ex. 2. Given that x is an integer, determine its value from the following inequalities:

$$\begin{cases} 4x - 7 < 2x + 3 \\ 3x + 1 > 13 - x \end{cases}$$

Transposing terms,

$$\begin{cases} 2x < 10 \\ 4x > 12 \end{cases}$$

Dividing by coefficient of x in each inequality,

$$\begin{cases} x < 5 \\ x > 3 \end{cases} \qquad \therefore x = 4 \textit{ Ans.}$$

Let the pupil illustrate this solution graphically.

160. Proof of Unconditional Inequalities.

Ex. 1. Prove that the sum of the squares of any two unequal quantities is greater than twice their product.

Let a be the greater of the two quantities, and b the less.

Then, $a - b > 0$

$$\therefore (a - b)^2 > 0$$

$$\therefore a^2 - 2ab + b^2 > 0$$

$$a^2 + b^2 > 2ab$$

Ex. 2. Prove $(a + b)(b + c)(a + c) > 8abc$.

The left-hand member when expanded becomes

$$a(b^2 + c^2) + b(a^2 + c^2) + c(a^2 + b^2) + 2abc$$

But from Ex. 1,

$$a(b^2 + c^2) > a(2bc) \quad \ldots\ldots\ldots \quad (1)$$

$$b(a^2 + c^2) > b(2ac) \quad \ldots\ldots\ldots \quad (2)$$

$$c(a^2 + b^2) > c(2ab) \quad \ldots\ldots\ldots \quad (3)$$

Also,

$$2abc = 2abc \quad \ldots\ldots\ldots \quad (4)$$

Adding (1), (2), (3), (4),

$$(a + b)(b + c)(a + c) > 8abc$$

EXERCISE 82

Reduce:

1. $(x + 1)^2 < x^2 + 3$

2. $(3 - x)^2 > (x - 4)^2$

3. $7ax + b > 3ax + 5b$

4. $\dfrac{4x-3}{3} > \dfrac{x}{2} + \dfrac{3x + 8}{21}$

5. $\dfrac{10-4x}{3} < x - \frac{11}{3} - 7$

6. $\dfrac{x + 1}{x - 2} > \dfrac{x + 3}{x - 4}$

7. $\dfrac{a + x}{a - x} > \dfrac{b + x}{2b - x}$

8. $\dfrac{4(x + 3)}{9} < \dfrac{8x + 37}{18} - \dfrac{7x - 29}{5x - 12}$

Find the limits of x:

9. $3x + 1 > 2x + 7$
$2x - 1 < x + 6$

10. $3(x-4)+2 > 4(x-3)$
$2(x+1) < 4(x-1)+3$

11. $60 > \dfrac{x - 1}{5} > 50$

12. $100 > x + \dfrac{x + 1}{2} > 90$

Solve the following:

13. $x - y > 5$
$x + y = 12$

14. $3x - 4y > 6$
$4x + 5y = 80$

15. What number is that whose fifth plus its sixth is greater than 6, while its third minus its eighth is less than 4?

16. A certain integer decreased by $\frac{2}{5}$ of itself is greater than $\frac{1}{6}$ of the number increased by $5\frac{1}{2}$; but if $\frac{1}{2}$ of itself is added to the number, the sum is less than 20. Find the number.

If the letters employed in each are positive and unequal, prove:

17. $3a^2 + b^2 > 2a(a + b)$

18. $a^3 - b^3 > 3a^2b - 3ab^2$

19. $a^3 + b^3 > a^2b + ab^2$

20. $\frac{a}{b} + \frac{b}{a} > 2$

21. $a + b > 2\sqrt{ab}$

22. $a^2 + b^2 + c^2 > ab + ac + bc$

23. $6abc < a(b^2 + ab + c^2) + c(b^2 + bc + a^2)$

24. $ab(a + b) + ac(a + c) + bc(b + c) < 2(a^3 + b^3 + c^3)$

25. $a^3 + b^3 + c^3 > 3abc$

26. A baseball team has won 22 games out of 35 games played. What is the least number of games which the team must win in succession in order that its average of games won may exceed .75?

27. Make up and work a similar example concerning games won by a basketball team.

28. A boy has worked correctly 13 examples out of 18. What is the least number of examples which he must work correctly in succession in order to bring his average above 90%? Above 80%?

29. Who invented the signs > and < to represent inequality? What other signs were invented for this purpose? Which set of signs do you consider superior and why?

CHAPTER XV

INVOLUTION AND EVOLUTION

Involution

161. Involution is the operation of raising an expression to any required power.

Since a power is the product of equal factors, involution is a species of multiplication. In this multiplication, the fact that the quantities multiplied are equal leads to important abbreviations of the work.

Powers of Monomials

162. Law of Exponents or Index Law.

Since, $a^3 = a \times a \times a,$

$$(a^3)^4 = (a \times a \times a)(a \times a \times a)(a \times a \times a)(a \times a \times a)$$
$$= a^{3\times 4} = a^{12}.$$

In general, in raising a^n to the m^{th} power, we have the factor a taken $m \times n$ times, or

$$(a^n)^m = a^{mn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

Also, $(ab)^n = ab \times ab \times ab \quad . \quad . \quad . \quad . \quad . \quad \text{to } n \text{ factors}$

$$= (a \times a \times a \ . \ . \ . \text{ to } n \text{ factors})\ (b \times b \times b \ . \ . \text{ to } n \text{ factors})$$

$$\therefore (ab)^n = a^n b^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

This law enables us to reduce the process of finding the power of a product to the simpler process of finding the power of each factor of the given product.

163. Law of Signs. It is evident from the law of signs in multiplication that

(1) *An even power of a quantity (whether plus or minus) is always positive.*

Thus, $(-3)^2 = 9$, and $(-ab)^4 = a^4b^4$.

(2) *An odd power of a quantity has the same sign as the original quantity.*

Thus, $(-a)^7 = -a^7$, and $(+a)^5 = a^5$.

164. Involution of Monomials in General. Hence, to raise a monomial to a required power,

Raise the coefficient to the required power;

Multiply the exponent of each literal factor by the index of the required power;

Prefix the proper sign to the result.

Ex. 1. Find the cube of $3x^2y$.

$$(3x^2y)^3 = 27x^6y^3 \quad \textit{Ans.}$$

Ex. 2. $(-2ab^3)^5 = -32a^5b^{15}$ *Ans.*

165. Powers of Fractions. By a method similar to that used in Art. 162, it can be shown that

$$\left(\frac{a^n}{b^m}\right)^p = \frac{a^{pn}}{b^{pm}}$$

Hence, to raise a fraction to a required power,

Raise both numerator and denominator to the required power, and prefix the proper sign to the resulting fraction.

$$\text{Ex.} \quad \left(-\frac{2a^3x}{5b^2y^3}\right)^4 = \frac{16a^{12}x^4}{625b^8y^{12}} \quad \textit{Ans.}$$

EXERCISE 83

Write the square of

1. $7a^2b$
2. $-5xy^3$
3. $\frac{2}{3}x^2y^5$
4. $-\dfrac{3x}{5z^2}$
5. $-\dfrac{6ab}{11}$

6. $-13x^ny$

7. $\dfrac{x^ny^m}{10z}$

8. $-4y^{n+1}$

9. $-\dfrac{1}{8cd}$

Write the cube of

10. $3xy$ **11.** $-2x^2$ **12.** $\frac{1}{2}x^3y^2$ **13.** $-5x^ny$

14. $-\dfrac{3x}{4y^2}$ **15.** $-\dfrac{5c^2d^5}{7x^{2n}}$

Write the value of

16. $(7ab^2c^3)^3$

17. $(11a^5b^4)^2$

18. $(\frac{3}{2}x^2y)^4$

19. $\left(\dfrac{2x^2y^2}{x^nz}\right)^5$

20. $(-\frac{1}{3}a^3x^n)^4$

21. $(-2x^3)^7$

22. $(-\frac{1}{2}m^2)^6$

23. $(3\frac{1}{2}x^{2n})^3$

24. $(.03)^2$

25. $(.003)^2$

26. $(.03)^3$

27. $(.003)^3$

28. $\left(-\dfrac{5x^n}{3a^{n-1}}\right)^4$

29. $\dfrac{1}{2}\left(\dfrac{2}{3}\right)^5$

30. $(2\frac{1}{2})^2$

31. Commit to memory the values of the various powers of

(1) 2 up to 2^{10}

(2) 3 up to 3^6

(3) 4 up to 4^5

(4) 5 up to 5^4

(5) 6 up to 6^4

(6) the squares of the numbers up to 29

32. Give the value of 2×3^2, $(2 \times 3)^2$.

33. Give the value of $\left(\dfrac{3}{5}\right)^2$, $\dfrac{3^2}{5}$, $\dfrac{3}{5^2}$.

34. Give the value of each of the following: $[(-2)^3]^2$, $[(-2)^2]^3$, $[(-3)^2]^2$. Does $[(-a)^3]^2$ equal $[(-a)^2]^3$?

35. Does 2×3^5 equal 6^5? Does $\frac{1}{2}(4^5)$ equal 2^5?

36. On squared paper show the meaning of $(.3)^2 = .09$. Also of $(1.5)^2$. Of $(1.5)^2$, $(2.5)^2$, $(.5)^2$.

37. Given $2^5 = 32$, find in the shortest way the value of 2^{10}. Also of 2^{15}. Of 2^{12}.

38. Make up and work an example similar to Ex. 37 concerning powers of 3. Of 4.

39. Give the value of $(-2)^{10}$. Of $(-2)^9$.

40. In $x^3 \cdot x^5$, how many x's are multiplied together? How many in $(x^3)^5$? Write out in full each of these sets of factors.

41. Treat in the same way $a^4 \cdot a^5$ and $(a^4)^5$.

42. Make up and work an example similar to Ex. 40.

43. Treat $[(x^2)^3]^4$ and $(x^2)^3 \cdot x^4$ in like manner.

44. In obtaining the value of $\frac{(2^3)^4}{2^3 \times 2^4}$, is it allowable to cancel the 4's?

45. In reducing $\frac{(x^3)^5}{x^3 \cdot x^5}$ to its simplest form, is it allowable to cancel the 5's? Why?

46. Is it allowable to cancel the 4's in obtaining the value of $\frac{4 \times 2^5}{4 \times 2^3}$? Of $\frac{2^4}{3^4}$?

47. Does $2^3 \times 2^4$ equal 4^7? Give a reason for your answer. The value of the second expression is how many times as great as that of the first?

48. Express as a power of 2 the number of great-grandparents a person has. Also the number of great-great-great-grandparents.

49. If a decimal fraction contains four places, how many places will its square contain? Its cube? Its fourth power? Give a numerical illustration of your answer to the first of these questions.

50. To $\frac{15b^2}{2a^2}$ add the square of $\frac{1}{2}$ of $\frac{b}{2a}$.

51. Find the value of ar^{n-1} when $a = \frac{1}{2}$, $r = \frac{2}{3}$, and $n = 6$.

In each of the following let $x = 0; \frac{1}{2}; 1; 2; 3; -\frac{1}{2}; -1; -2; -3$, in succession, and tabulate the results obtained as in Ex. 54, p. 24.

52. $x^2 - 3x + 2$

53. $x^2 - 2x - 1$

54. $x^3 + x$

55. $x^3 - x$

56. $x^3 - x^2 + 1$

57. $x^3 - 2x^2 - 2$

58. Show that $2^8 \times 5^8 = 10^8$. Is there any advantage in knowing that $2^8 \times 5^8 = 10^8$?

59. Work again such examples on p. 105 as the teacher may indicate.

60. How many of the examples in this Exercise can you work at sight?

Powers of Binomials

166. General Method. In obtaining a required power of a binomial, it is possible to abbreviate the work even more than in the involution of a monómial.

It is sufficient, in taking up the subject here for the first time, to obtain several powers of a binomial by actual multiplication, and, by comparing them, to obtain a general method for writing out the power of any binomial. A formal proof of the method is given later.

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

If b is negative, the terms containing odd powers of b will be negative; that is, the second, fourth, sixth, and all even terms, will be negative.

Comparing the results obtained, it is perceived that

I. The **number of terms** equals the exponent of the power of the binomial, plus one.

II. **Exponents.** The exponent of a in the first term equals the index of the required power, and diminishes by 1 in each succeeding term. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

III. **Coefficients.** The coefficient of the first term is 1; of the second term it is the index of the required power.

In each succeeding term the coefficient is found by *multiplying the coefficient of the preceding term by the exponent of* a *in that term, and dividing by the number of the preceding term.*

IV. **Signs of terms.** If the binomial is a difference, the signs of the even terms are minus; otherwise the signs of all the terms are plus.

Ex. $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$
$+ 21a^2b^5 + 7ab^6 + b^7.$

The coefficient of the third term $= \dfrac{7 \times 6}{2} = 21.$

The other coefficients are determined similarly.

Observe that the *coefficients of the latter half of the expansion are the same as those of the first half in reverse order.*

167. Binomials with Complex Terms. If the terms of the given binomial have coefficients or exponents other than unity, it is usually best to separate the process of writing out the required power into two steps.

Ex. $(2x^3 - \frac{1}{4}y^2)^4 = (2x^3)^4 - 4(2x^3)^3(\frac{1}{4}y^2) + 6(2x^3)^2(\frac{1}{4}y^2)^2$
$-4(2x^3)(\frac{1}{4}y^2)^3 + (\frac{1}{4}y^2)^4$
$= 16x^{12} - 8x^9y^2 + \frac{3}{2}x^6y^4 - \frac{1}{8}x^3y^6 + \frac{1}{256}y^8$

Let the pupil check the work by letting $x = 2$, $y = 2$.

168. Application to Polynomials.

Ex. $(x + 2y + 3z)^3 = [(x + 2y) + 3z]^3$

$$= (x + 2y)^3 + 3(x + 2y)^2(3z) + 3(x + 2y)(3z)^2 + (3z)^3$$

$$= x^3 + 6x^2y + 12xy^2 + 8y^3 + 9x^2z + 36xyz + 36y^2z + 27xz^2 + 54yz^2 + 27z^3 \quad \textit{Ans.}$$

Let the pupil check the work.

EXERCISE 84

Expand and check:

1. $(a + x)^5$
2. $(a - x)^5$
3. $(b + y)^7$
4. $(b - y)^6$
5. $(a - y)^9$
6. $(a + y)^8$
7. $(2a + b)^5$
8. $(2a + \frac{1}{2}b)^5$
9. $(1 - x)^6$
10. $(2c - d^2)^5$
11. $(7 - 3x^2)^3$
12. $(3 - \frac{1}{2}c^2)^5$
13. $(2x^2 - 1)^6$
14. $(2a^2 - 3b)^4$
15. $(x^2 + 2a)^5$
16. $(3a^2 + \frac{1}{3})^5$
17. $(2x^2 + \frac{1}{2})^8$
18. $(2 - \frac{x}{3})^7$
19. $(x^2 + x - 1)^3$
20. $(x^2 - 3x - 1)^3$
21. $(a^2 + ac + c^2)^4$
22. $(x - y + z)^3$
23. $(2x^2 - x + 3)^3$
24. $(1 + x - x^2)^4$

25. How many terms are there in the expansion of $(a+b)^2$? Of $(a + b)^3$? $(a + b)^n$?

26. Write out the last three terms in the expansion of $(a + b)^{15}$.

27. Who first suggested the writing of an exponent in its present position (that is, a little above and to the right of the base)? (See p. 456.) Tell all you can about this man.

28. Give some of the ways in which powers were previously indicated and exponents written. Discuss the relative merits of these different ways of writing exponents.

29. How many examples in Exercise 35 (p. 131) can you now work at sight?

EVOLUTION

169. **A Root** of a quantity is a quantity which, taken as a factor a given number of times, will produce the given quantity.

170. **Evolution** is the process of finding a required root of a quantity.

What is the radical or root sign? What is the meaning of $\sqrt{9}$? Of $\sqrt[3]{a}$? $\sqrt[n]{7}$?

171. **Number of Roots.** Taking a particular example, we find that $\sqrt{4}$ has two values, *viz.*: $+2$ and -2, for $(+2)^2=4$, and $(-2)^2=4$.

A number containing a square root of a negative quantity is termed an **imaginary number.**

A **real number** is a number which does not contain an imaginary number.

The nature of the square of an imaginary number, as of $\sqrt{-4}$, is explained in Chapter XVIII (p. 334).

If we include imaginary roots, it may be shown that when any root of a given number is extracted, *the number of possible roots equals the index of the root to be extracted.*

Thus, in taking the cube root of 8, we find three possible roots, *viz.*: 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

172. **The Principal Root** of a number is that real root of the number which has the same sign as the number itself.

Thus, the principal root for $\sqrt{4}$ is 2; for $\sqrt[3]{27}$ is 3; for $\sqrt[3]{-27}$ is -3.

In this chapter only the principal roots of numbers are considered.

EVOLUTION OF MONOMIALS

173. Index Law. Since $(a^m)^n = a^{mn}$ (Art. 162, p. 272), it follows that

$$\sqrt[n]{a^{mn}} = a^m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \text{I.}$$

where m and n are positive integers.

Hence, the process of finding the root of a quantity affected by an exponent becomes simply a division of exponents.

Also, $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ . . . . II.

For let $\sqrt[n]{a} = x, \qquad \sqrt[n]{b} = y;$

$$\therefore x^n = a \; . \; . \; . \; (1) \qquad y^n = b \; . \; . \; . \; (2)$$

But $x^n y^n = (xy)^n$ (by Art. 162)

Substitute for x^n and y^n from (1) and (2),

$$ab = (\sqrt[n]{a}\sqrt[n]{b})^n \quad . \; . \; . \; . \; . \; . \; . \; (3)$$

Extract the nth root of each member of (3),

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

This reduces the process of finding the nth root of a product to the simpler process of finding the nth root of each factor.

174. Method. Hence, to extract a required root of any monomial,

Extract the required root of the coefficient;

Divide the exponent of each letter by the index of the required root;

Prefix the proper sign to the result.

How may the work be checked?

EXERCISE 85

Write the square root of

1. $9x^2y^4$
2. $25a^6$
3. $144y^{2n}$
4. $16x^2y^2$
5. $\dfrac{36a^8}{49x^{10}}$
6. $\frac{4}{9}x^2y^{12}$
7. $\dfrac{121a^6x^{2m}}{81y^{2n+2}}$

Write the cube root of

8. $27x^6$

9. $125y^3z^9$

10. $-\frac{1}{8}a^3b^6$

11. $\dfrac{27a^{12}x^9}{64y^{3n}}$

12. $-\dfrac{216x^{27}}{343y^9}$

13. $\dfrac{8x^{15}y^3}{z^{3n+3}}$

14. $-\dfrac{1000}{x^{6n-9}}$

Write the value of

15. $\sqrt[3]{-512x^6}$

16. $\sqrt[4]{16y^{12}}$

17. $\sqrt[5]{-32x^{10}y^5}$

18. $\sqrt[6]{64a^{30}x^{6n}}$

19. $\sqrt[4]{\frac{1}{81}x^8y^{20}}$

20. $\sqrt[5]{\frac{32}{243}x^{25}}$

21. $\sqrt[4]{\dfrac{625x^{20}}{y^{4n+8}}}$

22. $\sqrt[5]{-\frac{1}{32}x^{30n}y^5}$

23. Express 32 as a power of 2. 81 as a power of 3.

24. Express each of the following as a power of some number (a prime number, if possible): 32, 243, 256, $\frac{27}{64}$, $\frac{32}{243}$, $\frac{121}{256}$, and 729.

25. Find the value of x in each of the following equations: $x^3 = 64$, $x^5 = \frac{32}{243}$, $x^3 = -8$, $x^7 = 128$.

26. If $2^x = 16$, what is the value of x?

SUG. Write $2^x = 2^4$.

27. Solve each of the following for x: $2^x = 32$, $3^x = 81$, $3^x = 243$, and $\left(\frac{2}{3}\right)^x = \frac{16}{81}$.

28. Find the value of r or n in each of the following: $2^r = 16$, $r^5 = 32$, $r^4 = \frac{16}{81}$, $-\frac{32}{243} = \left(\frac{2}{3}\right)^n$, and $2^{r-1} = 32$.

29. Find the largest square factor in each of the following numbers: 45, 128, 192, 112, and 147.

30. Find the largest cube that is a factor of 54. Of 81, 128, $\frac{81}{125}$. Also the largest 4th power that is a factor of 32. Of 162, 256.

Extract the square root of each of the following by taking out factors which are perfect squares:

31. 5625	**33.** 46,656	**35.** $48 \times 16 \times 18 \times 24$
32. 1296	**34.** 63×28	**36.** $27 \times 12 \times 98 \times 50$

37. Who first used the sign $\sqrt{}$ to denote a root? Who first suggested the use of the vinculum instead of the parenthesis (as in the latter part of $\sqrt{}$)? How were roots like $\sqrt[3]{}$, $\sqrt[4]{}$, first indicated after the invention of the radical sign?

38. Make up and work an example similar to Ex. 25. To Ex. 29. Ex. 33.

39. How many of the examples in this Exercise can you work at sight?

Square Root

175. Square Root of Polynomials. In order to determine a general method for finding the square root of any polynomial which is a square, we consider the relation between the terms of a binomial and the terms of its square; as between $a + b$ and its square, $a^2 + 2ab + b^2$. This relation stated in inverse form gives us the required method. The essence of the method consists in writing $a^2 + 2ab + b^2$ in the form $a^2 + b(2a + b)$.

Ex. Extract the square root of $16x^2 - 24xy + 9y^2$.

$$
\begin{array}{l|l|l}
 & 16x^2 - 24xy + 9y^2 & \underline{4x - 3y \ \textit{Root}} \\
 & 16x^2 & \\
\hline
8x - 3y & -24xy + 9y^2 & \\
 & \underline{-24xy + 9y^2} &
\end{array}
$$

Taking the square root of the first term, $16x^2$, we obtain $4x$, which is placed to the right of the given expression as the first term of the root. Subtract the square of $4x$ from the given polynomial.

Taking twice the first term of the root, $8x$, as a trial divisor, and dividing it into the first term of the remainder, we obtain the second term of the root, $-3y$. This is annexed to the first term of the root and also to the trial divisor to make the complete divisor, $8x - 3y$.

176. Square Root to Three or More Terms. In squaring a trinomial, $a + b + c$, we may regard $a + b$ as a single quantity, and denote it by a symbol, as p, and obtain the square in the form $p^2 + 2pc + c^2$.

Evidently we may reverse this process, and extract a square root to three terms by regarding two terms of the root, when found, as a single quantity. So a fourth term of a root, or any number of terms, may be obtained by regarding the root already found as a single quantity.

Ex. Extract the square root of $x^4 - 6x^3 + 19x^2 - 30x + 25$.

$$\begin{array}{r|l|l} & x^4 - 6x^3 + 19x^2 - 30x + 25 & \underline{x^2 - 3x + 5} \textit{ Root} \\ & x^4 & \\ \hline 2x^2 - 3x & -6x^3 + 19x^2 & \\ -3x & -6x^3 + 9x^2 & \\ \hline 2x^2 - 6x + 5 & +10x^2 - 30x + 25 & \\ & +10x^2 - 30x + 25 & \\ \hline \end{array}$$

The first two terms of the root, $x^2 - 3x$, are found as in the example in Art. 175.

To continue the process, we consider the root already found, $x^2 - 3x$, as a single quantity, and multiply it by 2 to make it a trial divisor.

Dividing the first term of the remainder, $10x^2$, by the first term of the trial divisor, $+2x^2$, we obtain the next term of the root, $+5$.

The process is then continued as before.

The work may be checked by squaring the result obtained, or by numerical substitution.

Let the pupil state the above process as a rule.

EXERCISE 86

Find the square root and check:

1. $x^4 - 4x^3 + 6x^2 - 4x + 1$.

2. $1 - 2a - a^2 + 2a^3 + a^4$.

3. $9x^4 - 12x^3 + 10x^2 - 4x + 1$.

4. $25 + 30x + 19x^2 + 6x^3 + x^4$.

5. $n^6 - 4n^5 + 4n^4 + 6n^3 - 12n^2 + 9$.

6. $4x^6 + 12x^5 + x^4 - 24x^3 - 14x^2 + 12x + 9$.

7. $1 + 16m^6 - 40m^4 + 10m - 8m^3 + 25m^2$.

8. $46n^2 + 25n^4 + 4n^6 + 25 - 44n^3 - 40n - 12n^5$.

9. $9x^6 + 9y^6 + 24x^5y + 24xy^5 - 8x^4y^2 - 8x^2y^4 - 50x^3y^3$.

10. $m^2 + 9 + x^2 + 6m + 6x + 2mx$.

11. $1 + 5x^2 + 2x^4 + x^6 - 4x^5 + 2x^3 + 2x$.

12. $28x^2 - 47x^4 + 49x^6 - 42x^5 - 4x^3 + 16x + 4$.

13. $\frac{1}{4}x^2 - 5x + 25$.

14. $\frac{9}{4}x^2 - 5xy + \frac{25}{9}y^2$.

15. $\dfrac{x^2}{4y^2} - \dfrac{2x}{y} + 4$.

16. $x^4 + 2x^3 - x + \frac{1}{4}$.

17. $\frac{1}{4}a^4 - \frac{1}{3}a^3 + \frac{55}{9}a^2 - 4a + 36$.

18. $\dfrac{a^2}{x^2} + \dfrac{6a}{x} + 11 + \dfrac{x^2}{a^2} + \dfrac{6x}{a}$.

19. $\frac{4}{9}x^4 - \frac{4}{5}x^3 + \frac{277}{75}x^2 - 3x + \frac{25}{4}$.

20. $\frac{9}{25}x^4 - \frac{2}{5}x^3 + \frac{29}{18}x^2 - \frac{5}{6}x + \frac{25}{16}$.

21. $1 + a - \frac{5}{12}a^2 - \frac{7}{3}a^3 - \frac{8}{9}a^4 + \frac{2}{3}a^5 + a^6$.

22. $\dfrac{c^4}{4} + cx + \dfrac{x^2}{c^2} + \dfrac{1}{4x^2} + \dfrac{c^2}{2x} + \dfrac{1}{c}$.

23. $\dfrac{a^2}{x^2} - ax + \dfrac{x^2}{a^2} - 2 + \dfrac{a^3}{x} + \dfrac{a^4}{4}$.

Find to three terms the square root of

24. $1 + 4x$.

25. $x^2 - 6$.

26. $a^2 + 4b$.

27. $4a^2 - 6ab$.

28. $x^2 + 3$.

29. $a^2 + 3ab - 2b^2$.

30. Solve $x^2 - 6x + 9 = 25$.

Extracting the square root of each number we have $x - 3 = \pm 5$ whence $x = 3 \pm 5$, and $x = 8$, or -2.

Let the pupil check these results.

Solve the following equations:

31. $x^2 - 6x + 9 = 36$

32. $x^2 + 4x + 4 = 25$

33. $x^2 + 8x + 16 = 1$

34. $x^2 + 8x = -15$

35. $4x^2 + 12x + 9 = 16$

36. $16x^2 - 8x + 1 = 4$

37. $y^2 - 2y + 1 = 9$

38. $r^2 + 10r + 25 = 16$

39. Make up and work an example similar to Ex. 3. To Ex. 26. Ex. 34.

40. How many examples in Exercise 45 (p. 155) can you now work at sight?

177. Square Root of Arithmetical Numbers. The same general method as that used in Art. 175 may be used to extract the square root of arithmetical numbers.

The details of the method of extracting the square root of numbers are explained in arithmetic (see Durell's *Advanced Arithmetic*). As illustrations of the process, we give the following examples:

Ex. 1. Extract the square root of 1849.

$$\begin{array}{rr|l|l}
 & & \overset{\frown}{18}\overset{\frown}{49} & \underline{40 + 3,} \\
40^2 = & & 1600 & \\
\cline{3-3}
2 \times 40 = & 80 & 249 & \\
 & 3 & 249 & \\
\cline{2-3}
 & 83 & &
\end{array}
\qquad \text{or more briefly,} \qquad
\begin{array}{r|l|l}
 & \overset{\frown}{18}\overset{\frown}{49} & \underline{43} \; \textit{Root} \\
 & 16 & \\
\cline{2-2}
83 & 249 & \\
 & 249 & \\
\cline{2-2}
\end{array}$$

Let the pupil check the work.

Ex. 2. Extract the square root of 18.550249.

$$\begin{array}{r|l|l}
 & \overset{\frown}{18}.\overset{\frown}{55}\overset{\frown}{02}\overset{\frown}{49} & \underline{4.307} \; \textit{Root} \\
 & 16 & \\
\cline{2-2}
83 & 255 & \\
 & 249 & \\
\cline{2-2}
8607 & 60249 & \\
 & 60249 & \\
\cline{2-2}
\end{array}$$

Ex. 3. Extract the square root of $\frac{2}{3}$.

$$\tfrac{2}{3} = .66666666+$$

Extracting the square root of .66666666+, we obtain .8164+ as the desired result.

178. For extraction of **Cube and Higher Roots,** see Appendix, p. 472.

EXERCISE 87

Find the square root and check where possible:

1. 7225
2 2601
3. 105625
4. 182329
5. 337561
6. 567009
7. 36144144
8. 8114.4064
9. 199.204996
10. 10.30731025
11. .0291419041
12. 1513689.763041

Find to four decimal places the square root of

13. 7
14. 11
15. 12.5
16. $3\frac{1}{3}$
17. $2\frac{1}{9}$
18. .9
19. $6\frac{2}{3}$
20. $1\frac{5}{8}$
21. $\frac{7}{60}$
22. .049
23. 1.0064
24. $36\frac{1}{11}$

Compute to three decimal places the value of

25. $\sqrt{2+\sqrt{3}}$
26. $\sqrt{\sqrt{13}-\sqrt{3}}$
27. $\sqrt{3\sqrt{3}+\sqrt{5}}$
28. $\sqrt{3\sqrt{6}-2\sqrt{7}}$
29. $\sqrt{\dfrac{\sqrt{5}-\sqrt{2}}{4}}$
30. $\sqrt{\dfrac{5(\sqrt{3}-\sqrt{2})}{2}}$
31. $\sqrt{\dfrac{7\sqrt{3}+2\sqrt{5}}{4}}$

32. Find the diagonal of a square whose side is 5 in.

33. Find the side of a square whose diagonal is 5 in.

34. If a city park is $\frac{1}{2}$ mi. long and 300 yd. wide, how much

is saved by walking from a corner to the opposite corner along the diagonal instead of along the sides?

35. In $K = \sqrt{s(s-a)(s-b)(s-c)}$, if $a = 25$, $b = 63$, $c = 74$, and $s = \frac{1}{2}(a + b + c)$, find K.

Do you know what figure the above formula gives the area of?

36. Show the meaning of $\sqrt{.09} = .3$ on squared paper. Also of $\sqrt{.009}$ as well as you can.

37. If $4\pi R^2 = 10$ and $\pi = \frac{22}{7}$, find R.

38. On squared paper construct the triangle whose vertices are the points (3, 0), (1, 5), and (−2, −3), and find the length of its sides.

39. The area of Texas is 265,780 sq. mi. Find the side of a square having an equivalent area. Think of some distance familiar to you which is approximately equal to a side of this square, then picture to yourself the whole square and thus visualize the area of Texas.

40. By the method of Ex. 39 visualize the area of the state in which you live.

41. In like manner visualize the area of Germany, which is 208,830 sq. mi.

42. The population of New York City in the year 1910 was 4,766,883. What is the side of a square which would be covered by this number of people if each person occupied a space $30'' \times 15''$? Hence visualize the population of New York City as it was at this date.

43. By the method of Ex. 42 visualize the population of a large city in your neighborhood.

44. In like manner visualize 30,000 sheep herded together.

45. In a recent year the record yield of corn for an acre was 255 bu. If a bushel is taken as $1\frac{1}{4}$ cu. ft., find the side of a square bin 3 ft. deep which would just hold this corn.

46. In a recent year the record yield of milk for one cow was 27,432 lb. If $7\frac{1}{2}$ lb. make a gallon and a gallon is 231 cu. in., find the side of a square tank 2 ft. deep which would just hold this milk.

47. A baseball dropped from the top of the Washington Monument has been caught by a player standing on the ground at the foot of the monument. The velocity attained by a falling body is given by the formula $v = \sqrt{2gs}$, where v denotes the velocity attained, s the distance through which the body falls, and $g = 32.16$ ft. The height of the monument is 555 ft. Find the velocity of the ball when caught.

48. A stone is dropped from a balloon a mile above the earth. Find the velocity with which the stone strikes the earth (resistance of the air being neglected).

At sight give the value of

49. $2 + \sqrt{9}$

50. $\sqrt{16} - \sqrt{9}$

51. $3\sqrt{16} - 2\sqrt{9}$

52. $\sqrt{25} \times \sqrt{16}$

53. $\sqrt{64} \div \sqrt{4}$

54. $10 - \sqrt{16}$

55. $9 \div \sqrt[3]{27}$

56. $2\sqrt[3]{125} - 3\sqrt[3]{64}$

57. $3\sqrt{9} - 2\sqrt{8}$

58. $\dfrac{2 + 4\sqrt{25}}{\sqrt{4}}$

59. $\dfrac{3\sqrt{64} - \sqrt{9}}{3}$

60. Make up and work an example similar to Ex. 21. To Ex. 39. Ex. 53.

CHAPTER XVI

EXPONENTS

179. Positive Integral Exponents. Using a^3 as a brief symbol for $a \times a \times a$, and a^m as a brief symbol for $a \times a \times a \times a \ldots\ldots$ to m factors, we have already found the following laws to govern the use of positive integral exponents:

I. $a^m \times a^n = a^{m+n}$

II. $\dfrac{a^m}{a^n} = a^{m-n}$, if $m > n$

III. $(a^m)^n = a^{mn}$

IV. $\sqrt[n]{a^{mn}} = a^m$

V. $(ab)^n = a^n b^n$

180. Fractional and Negative Exponents. We have seen that by using fractions as well as integers, and negative as well as positive quantity, the field of quantity and operation in algebra is greatly extended and some processes are made simpler, others more powerful. These same advantages are secured by the use of fractional and negative exponents.

Let us suppose that the first and fundamental Index Law, $a^m \times a^n = a^{m+n}$, holds for fractional and negative exponents, and then inquire what meaning must be assigned to these exponents.

We limit the fractional and negative exponents here treated to those whose terms are either positive or negative integers, and commensurable; that is, expressible in terms of the unit of quantity used in the given problem.

Exponents like $\sqrt{2}$, as in $a^{\sqrt{2}}$, are not included in the discussion, though the student will find later that the same laws hold for these exponents.

181. I. Meaning of a Fractional Exponent.

Since by Index Law I,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2,$$

it follows that $a^{\frac{2}{3}}$ is one of the three equal factors which may be considered as composing a^2; that is, $a^{\frac{2}{3}}$ is the cube root of a^2.

$$\therefore a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

So, in general,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots\ldots \text{ to } q \text{ factors,}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\ldots \text{ to } q \text{ terms}}$$

$$= a^{\frac{p}{q}\times q} = a^p$$

Hence, in general, *in a fractional exponent the numerator denotes the power of the base that is to be taken, and the denominator denotes the root that is to be extracted.*

Ex. 1. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ *Ans.*

Ex. 2. $a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3}{4}+\frac{1}{2}+\frac{1}{3}} = a^{\frac{19}{12}}$ *Ans.*

Ex. 3. $\sqrt{2}x^{a+b} \cdot 2^{\frac{3}{2}}x^{a-b} = 2^{\frac{1}{2}}x^{a+b} \cdot 2^{\frac{3}{2}}x^{a-b} = 2^2x^{2a} = 4x^{2a}$ *Ans.*

Ex. 4. $32^{\frac{6}{5}} = \sqrt[5]{32^6} = 2^6 = 64$ *Ans.*

Note that in Ex. 4 it is best to *extract the required root first.*

In the examples which involve letters, the work may often be checked by numerical substitutions.

EXERCISE 88

Express with radical signs:

1. $a^{\frac{1}{2}}$.

2. $c^{\frac{5}{3}}$.

3. $2a^{\frac{1}{4}}$.

4. $2a^2b^{\frac{2}{7}}$.

5. $5x^{\frac{2}{3}}y^{\frac{3}{4}}$.

6. $2c^{\frac{3}{2}}d^{\frac{4}{5}}$.

7. $ax^{\frac{3}{n}}$.

8. $x^{\frac{n}{m}}y^{\frac{m}{2n}}$.

Express with fractional exponents:

9. $\sqrt[3]{a^4}$. **12.** $a\sqrt{x}$. **15.** $\sqrt{x}\sqrt[3]{y^2}$. **18.** $\dfrac{3a\sqrt{x^3}}{4\sqrt[3]{a}\sqrt{c^3}}$.

10. $\sqrt{c^5}$. **13.** $b\sqrt[3]{y}$. **16.** $2\sqrt[4]{x^5}\sqrt{y}$. **19.** $\dfrac{2\sqrt{81}\sqrt{3^4}}{3\sqrt{2^3}\sqrt{2^2}}$.

11. $2\sqrt[7]{x^2}$. **14.** $2x\sqrt{y^5}$. **17.** $\sqrt[3]{5}\sqrt[7]{a^{10}}$.

Find the value of

20. $27^{\frac{2}{3}}$. **23.** $\sqrt[3]{64^2}$. **26.** $(-27)^{\frac{4}{3}}$. **29.** $(-243)^{\frac{3}{5}}$.

21. $25^{\frac{3}{2}}$. **24.** $\sqrt[6]{64^7}$. **27.** $(-32)^{\frac{6}{5}}$. **30.** $(\frac{8}{27})^{\frac{4}{3}}$.

22. $16^{\frac{3}{4}}$. **25.** $(-8)^{\frac{2}{3}}$. **28.** $(-216)^{\frac{2}{3}}$. **31.** $(\frac{81}{16})^{\frac{5}{4}}$.

Simplify the following by performing the indicated operations:

32. $a^{\frac{1}{2}} \times a^{\frac{2}{3}}$. **35.** $2^{\frac{1}{2}}x^{\frac{1}{3}} \times 2^{\frac{3}{2}}x^{\frac{5}{3}}$. **38.** $\sqrt[3]{2}\sqrt{2^3}$.

33. $2a \times a^{\frac{1}{3}}$. **36.** $a^{\frac{4}{3}}\sqrt{a^3}$. **39.** $a^{\frac{3}{5}}\sqrt[4]{x^3} \cdot x^{\frac{1}{2}}\sqrt{a}$.

34. $a^2x^{\frac{1}{2}} \times a^{\frac{1}{2}}x^{\frac{3}{2}}$. **37.** $7\sqrt{a}\sqrt[3]{a^2}$. **40.** $x^{\frac{3}{5}}\sqrt[4]{x^5} \cdot 2x^{\frac{3}{20}}$.

41. $2^{\frac{3}{2}} \cdot 2^{\frac{5}{2}} \cdot 2^4$. **44.** $a^{2n-1} \cdot a^{2n+1}$.

42. $x^{a+b} \cdot x^{a-b}$. **45.** $x^{2(a+b)} \cdot x^{2(a-b)}$.

43. $y^{2p+3q} \cdot y^{2p-3q}$. **46.** $x^{p+q} \cdot x^{2q-1} \cdot x^{1-p}$.

47. $x^{3m-n} \cdot x^{2n+1} \cdot x^{m+2} \div x^{2m+n-3}$.

48. $\dfrac{a^{\frac{1}{2}}\sqrt{x^5}}{\sqrt[4]{c^3}} \times \dfrac{2x^{\frac{3}{2}}\sqrt{a^9}}{\sqrt[4]{c}}$. **49.** $\dfrac{\sqrt{2}\sqrt[5]{7}}{\sqrt[3]{3}\sqrt[4]{5}} \times \dfrac{2^{\frac{3}{2}}7^{\frac{4}{5}}}{\sqrt[3]{3^1}\sqrt[8]{5^{14}}}$.

50. Find the value of $5^{\frac{1}{2}}$ to three decimal places. Also of $5^{\frac{3}{2}}$ or $\sqrt{125}$. Multiply the two results. Compare the amount of work in this process with that of finding the value of 5^2. Which process gives the more accurate result?

51. What two parts are there to every power? What is the difference between an exponent and a power?

52. Make up and work an example similar to Ex. 34. To Ex. 49. Ex. 50.

53. How many of the examples in this Exercise can you now work at sight?

54. How many examples in Exercise 1 (p. 8) can you now work at sight?

182. II. Meaning of the Exponent Zero, or of a^0.

By direct division, $$\frac{a^m}{a^m} = 1$$

By subtraction of exponents, $$\frac{a^m}{a^m} = a^0 \qquad \therefore a^0 = 1$$

Thus, a^0 may be regarded as the result of dividing some power of a by itself.

An expression like px^2+qx+r is sometimes written $px^2+qx+rx^0$, the advantage being that in the latter form every term contains an x.

183. III. Meaning of a Negative Exponent.

By subtraction of exponents, $$\frac{a^k}{a^{k+n}} = a^{-n}$$

By cancellation, $$\frac{a^k}{a^{k+n}} = \frac{a^k}{a^k \times a^n} = \frac{1}{a^n}$$

$$\therefore a^{-n} = \frac{1}{a^n}$$

Ex. 1. $$4^{-2} = \frac{1}{4^2} = \frac{1}{16} \textit{ Ans.}$$

Ex. 2. $$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{4} \textit{ Ans.}$$

Negative exponents are useful in enabling us to write certain decimal fractions in an abbreviated form.

Ex. 3. Express .0000002 in a briefer form by the use of negative exponents.

$$.0000002 = \frac{2}{10{,}000{,}000} = \frac{2}{10^7} = 2 \times 10^{-7}$$

184. Transference of Factors in Terms of a Fraction. It follows from the meaning of a negative exponent that *any factor may be transferred from the numerator to the denominator of a fraction, or vice versa, provided the sign of the exponent of the factor is changed.*

Ex. 1. Transfer to the numerator the factors of the denominator of $\frac{ab^{-2}}{xy^{-\frac{1}{2}}}$.

$$\frac{ab^{-2}}{xy^{-\frac{1}{2}}} = ab^{-2}x^{-1}y^{\frac{1}{2}} \quad Ans.$$

Ex. 2. Express with positive exponents $\frac{2a^{-2}b}{xy^{-\frac{2}{3}}z^{-1}}$.

$$\frac{2a^{-2}b}{xy^{-\frac{2}{3}}z^{-1}} = \frac{2by^{\frac{2}{3}}z}{a^2x} \quad Ans.$$

EXERCISE 89

Transfer to the numerator all factors of the denominator:

1. $\frac{a}{c^2}$

2. $\frac{ab^3}{cd^{-2}}$

3. $\frac{3a}{2c^2}$

4. $\frac{1}{2^{-1}x^{\frac{1}{3}}y^{-4}}$

Express with positive exponents:

5. $7x^{-2}$

6. $\frac{x^{-3}y^{-1}}{3a^{-2}}$

7. $5a^{-1}b^3$

8. $\frac{5a^{-2}c^{\frac{1}{3}}}{2^{-1}xy^{-\frac{2}{3}}}$

9. $\frac{5a^2x^{-3}}{z^4}$

10. $\frac{3^{-2}\sqrt{a^{-1}}\sqrt[3]{x^2}}{2\sqrt{a^3}\sqrt[3]{x^{-1}}}$

Find the numerical value of

11. $4^{-\frac{1}{2}}$

12. $27^{-\frac{2}{3}}$

13. $\sqrt{\dfrac{1}{4^{-5}}}$

14. $\dfrac{1}{5^{-2}}$

15. $8^{-\frac{2}{3}}$

16. $\sqrt{81^{-3}}$

17. $\dfrac{1^{-\frac{2}{3}}}{3^{-3}}$

18. $3^{-2} \times 4^{\frac{1}{2}}$

19. $2^{-3} \div 8^{-\frac{1}{3}}$

20. $\left(\dfrac{1}{36}\right)^{-\frac{3}{2}}$

21. $\left(-\dfrac{8}{27}\right)^{-\frac{4}{3}}$

22. $\left(-\dfrac{8}{27}\right)^{-\frac{1}{3}}$

23. $(-125)^{-\frac{4}{3}}$

24. $\dfrac{(-8)^{-\frac{5}{3}}}{(-27)^{-\frac{2}{3}}}$

25 $(5\tfrac{4}{9})^{-\frac{3}{2}}$

26. $\left(\dfrac{1}{64}\right)^{\frac{5}{6}} \cdot \left(\dfrac{1}{4}\right)^{-\frac{5}{2}}$

27. $\dfrac{2^{-3} . 3^{-2} . 4^{-\frac{3}{2}}}{9^{-1} . 8^{-\frac{5}{3}}}$

28. Express .000,000,003 in a briefer form by the use of negative exponents. How many more figures and symbols are there in the first form than in the second?

29. Treat .000,000,001 in a briefer form by the use of negative exponents.

30. Show that 1 millimeter equals 10^{-3} meter.

31. The micron is a small unit of measure equal to one millionth of a meter. Express it as a part of a meter by use of a negative exponent.

32. The length of a wave of violet light is .000,016 in. Express this number as 16 in. multiplied by a power of 10.

Give the value of

33. $8^0,\ 80^0,\ \left(\dfrac{1}{8}\right)^0,\ \left(\dfrac{a}{b}\right)^0,\ (-8)^0,\ \left(-\dfrac{1}{8}\right)^0,\ \dfrac{1}{2^0},\ \dfrac{2^0}{1}.$

34. $3 \times 2^0,\ 3a^0,\ 3(\tfrac{1}{2})^0,\ 4^{\frac{1}{2}} \cdot 5^0,\ 4^{\frac{1}{2}} \div 5^0,\ \dfrac{6^0}{5^0}.$

35. $1^{\frac{3}{2}},\ 1^{-\frac{3}{2}},\ \$1^0,\ 1^{-\frac{2}{5}},\ 1^{-\frac{3}{2}} \times 4^{-\frac{1}{2}},\ 2^0 \times 1 - n.$

36. $8^{\frac{2}{3}} \times 8^{-\frac{1}{2}} \times 8^{\frac{5}{6}}.$

37. $16^{\frac{3}{4}} \times 16^{-\frac{1}{2}} \times 8^{\frac{2}{3}} \times 8^0.$

38. Express 4^0 as some power of 4 divided by itself.

39. Express x^0 as some power of x divided by itself.

40. Express 4^{-2} as the quotient of two powers of 4.

Express x^{-3} as the quotient of two powers of x. Express x^{-m} in like manner.

41. State the value of \$$4^0$. Of \$$4^{-2}$. \$$4^{-\frac{3}{2}}$.

42. Which is greater, $(\frac{1}{4})^{-2}$ or $(\frac{1}{4})^{-3}$?

Simplify the following by performing the indicated operations, and reducing the results:

43. $5a^{-7} \times 2a^5x^2$.

44. $4x^{-2} \div 2x^{-3}$.

45. $6a^{\frac{1}{2}}x^{-\frac{2}{3}} \cdot a^{\frac{1}{2}}x^{\frac{1}{3}}$.

46. $a^{-\frac{1}{3}} \cdot 2a^{\frac{1}{3}}$.

47. $8x^{-\frac{2}{3}}y \div 4x^{\frac{1}{3}}y^2$.

48. $7a^3x^{-\frac{3}{2}} \div 5x^{\frac{2}{3}}y$.

49. $x^{-\frac{3}{2}}\sqrt[4]{y^{-3}} \div x^{\frac{1}{2}}y^{\frac{1}{4}}$.

50. $\dfrac{x^{\frac{1}{2}}\sqrt[6]{x^{-7}}}{x^{-\frac{1}{3}}\sqrt{x^{-5}}}$.

51. $\dfrac{x^{\frac{1}{2}}\sqrt[3]{x^{-2}}}{\sqrt[6]{x^{-1}}}$.

52. $\dfrac{a\sqrt{a^{-2}}}{3\sqrt[3]{a^{-1}}}$.

53. $\dfrac{x^{a+b}x^{a-b}}{x^{2a}}$.

54. $\dfrac{7a^5\sqrt[3]{x^{-5}}}{3a^{-5}\sqrt[3]{x^{-4}}}$.

55. $\dfrac{x^n\sqrt{y^{-m}}}{y^m\sqrt{x^{-2n}}}$.

56. $\dfrac{x^{\frac{n}{2}}\sqrt[n]{y^{-2}}}{y^{\frac{1}{n}}\sqrt{x^{-n}}}$.

57. $\dfrac{\sqrt[2n]{x^3} \cdot \sqrt[n]{x^{-2}}}{x\sqrt[n]{x^{-3}}}$.

58. State which of the following has the greatest value: $(\frac{1}{2})^3$, $(\frac{1}{3})^{-2}$, $(\frac{1}{3})^0$, $\frac{1}{3}$.

Give the value of

59. $4^{-\frac{3}{2}} + 8^{\frac{1}{3}} + 8^0$.

60. $5^0 + 4^{-2} - (\frac{1}{8})^{-\frac{2}{3}}$.

61. $3^2 - 5 \times 4^0 + 8^{-\frac{2}{3}} + 1^{\frac{2}{3}}$.

62. $4^0 + x^0 - 5 \times 4^{\frac{3}{2}}$.

63. Make up and work an example similar to Ex. 21. To Ex. 28. Ex. 41. Ex. 59.

64. How many of the examples in this Exercise can you work at sight?

65. How many examples in Exercise 51 (p. 172) can you now work at sight?

185. Meaning of $(a^m)^n$ for Fractional and Negative Exponents. We now extend the law $(a^m)^n = a^{mn}$ to fractional and negative exponents.

Ex. 1. Find the value of $(4^{-\frac{3}{2}})^{-\frac{5}{3}}$.

$$(4^{-\frac{3}{2}})^{-\frac{5}{3}} = 4^{\frac{5}{2}} = 32 \textit{ Ans.}$$

Ex. 2. $$\sqrt[3]{\left(8a^{-\frac{5}{6}}\right)^2} = (8a^{-\frac{5}{6}})^{\frac{2}{3}} = 8^{\frac{2}{3}}a^{-\frac{5}{9}} = \frac{4}{a^{\frac{5}{9}}} \textit{ Ans.}$$

Ex. 3. $$\left(\frac{16a^{-4}}{81b^3}\right)^{-\frac{3}{4}} = \frac{16^{-\frac{3}{4}}a^3}{81^{-\frac{3}{4}}b^{-\frac{9}{4}}} = \frac{81^{\frac{3}{4}}a^3b^{\frac{9}{4}}}{16^{\frac{3}{4}}}$$

$$= \frac{27a^3b^{\frac{9}{4}}}{8} \textit{ Ans.}$$

Hence, in general, to simplify a complex expression in exponents,

Convert each radical sign into a fractional exponent;

Convert each power of a power into a power with a single exponent;

Convert each negative exponent into a positive exponent;

Simplify by cancellations and collections.

EXERCISE 90

Reduce to the simplest form:

1. $(a^2)^{-3}$. **3.** $(a^{-3})^{\frac{1}{2}}$. **5.** $(c^{\frac{1}{2}})^{-\frac{2}{3}}$.

2. $(x^{-2})^{-1}$. **4.** $(3^2)^0$. **6.** $(a^{-\frac{5}{3}})^{-\frac{3}{5}}$.

7. $x^{n+1} \cdot x^{n-1}$.

8. $(x^{n+1})^{n-1}$.

9. $(x^{-\frac{1}{3}}y^{\frac{1}{2}})^{-6}$.

10. $(64^{-1})^{-\frac{2}{3}}$.

11. $(9^{\frac{5}{2}})^{-\frac{3}{5}}$.

12. $(3a^{-\frac{2}{3}})^0$.

13. $(5x^{-\frac{1}{3}})^{-2}$.

14. $(x^{a+b} \cdot x^{a-b})^0$.

15. $(4x^{-4})^{-\frac{3}{2}}$.

16. $(9a^{-2}x^{\frac{1}{2}}y^{-3})^{-\frac{3}{2}}$.

17. $(-2a^2x^{-\frac{1}{2}})^2$.

18. $(-5x^{-1}y^{\frac{1}{2}})^{-2}$.

19. $(9x^{-3}y^{-2})^{-\frac{5}{2}}$.

20. $\dfrac{a^{n(n-1)}}{a^n}$.

21. $(a^{\frac{2}{3}}\sqrt[4]{a^{-3}})^{-\frac{6}{5}}$.

22. $\sqrt[6]{(a^{-1})^{-3}}$.

23. $(a^2\sqrt{3a^{-1}b^3})^{-2}$.

24. $\left\{\sqrt{(\sqrt{a^{-5}})^{-\frac{5}{2}}}\right\}^{-\frac{6}{5}}$.

25. $(c^{2a}x^{-3a})^{-\frac{b}{a}}$.

26. $\left(\dfrac{25\sqrt{x}}{9\sqrt[3]{x^{-2}}}\right)^{-\frac{3}{2}}$.

27. $\left\{\dfrac{27x\sqrt[3]{y^{-2}}}{8y^{\frac{1}{3}}\sqrt{x^{-2}}}\right\}^{-\frac{2}{3}}$.

28. $\left\{\sqrt[4]{x^{-\frac{2}{3}}y^{\frac{1}{2}}\sqrt{xy}}\right\}^{-\frac{2}{3}}$.

29. $\dfrac{\sqrt[3]{x^{-1}\sqrt{y^3}}}{\sqrt{y^{-1}\sqrt[3]{x^2}}}$.

30. $\dfrac{\sqrt[4]{8a^{-2}b^3\sqrt[3]{c^2}}}{\sqrt[3]{8^{-\frac{1}{4}}\sqrt{ab^{\frac{9}{2}}c^{-5}}}}$.

31. $\sqrt{x^{-1}\sqrt[3]{y^2}} \div \sqrt[3]{y\sqrt{x^{-3}}}$.

32. $\left(\dfrac{x^{-1}y^2}{a^3b^{-4}}\right)^{-2} \div \left(\dfrac{xy^{-1}}{a^{-2}b^3}\right)^3$.

33. $\left(\dfrac{x\sqrt[4]{a}}{b^2\sqrt{y^{-1}}} \times \dfrac{\sqrt[4]{a^3y^6}}{x^{-1}b^{-1}}\right)^{-2}$.

34. $\sqrt[4]{\sqrt[3]{(16^{-2})^6}}$.

35. $\dfrac{(x^{n-1})^n}{x^n \cdot x^{n-1}}$.

36. $\left[\sqrt[4]{\left(-\frac{27}{64}\right)^{-\frac{2}{3}}}\right]^{-\frac{2}{5}}$.

37. $\left(\dfrac{x^{n+1}}{x}\right)^n \div \left(\dfrac{x}{x^{1-n}}\right)^{n-1}$.

38. $8^{-\frac{2}{3}} + 9^{\frac{3}{2}} - 2^{-2} + 1^{-\frac{2}{5}} - 7^{\circ}$.

39. $\left\{\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{b^{-1}}} \cdot \dfrac{\sqrt{c^{-3}}}{a^{\frac{1}{3}}} \cdot \dfrac{b^{-\frac{1}{4}}\sqrt[3]{a}}{c^{-1}}\right\}^{-6}$.

40. $\left(a^{\frac{3}{4}}b^{-\frac{1}{2}}\sqrt[3]{a^{-2}b^2\sqrt{c^6}}\right)^6$.

41. $\left(\dfrac{a^{-4}}{b^{-2}c}\right)^{-\frac{3}{4}} \times \left(\dfrac{a^{-1}b\sqrt{c^{-3}}}{ab^{-1}}\right)^{\frac{1}{2}}$.

42. $\left\{\sqrt[5]{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}}\times\sqrt{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right\}^{-4}.$

43. $\left(\dfrac{a^2b}{c^2d}\right)^{\frac{1}{2}}\times\left(\dfrac{c^3d}{ab^3}\right)^{\frac{1}{3}}\times\left(\dfrac{a^{\frac{1}{3}}c}{b^{\frac{1}{4}}d^{\frac{5}{12}}}\right)^2.$

44. $\sqrt[5]{\dfrac{x^{\frac{5}{2}}y^{\frac{4}{3}}}{z^{-\frac{5}{4}}}\cdot\dfrac{z^4}{x^{-3}y^{-\frac{5}{3}}}\div\dfrac{y^{-2}z^{\frac{1}{4}}}{x^{-\frac{1}{2}}}}.$

45. $\dfrac{(x^n)^{n+2}}{(x^{n+1})(x^{n-1})}.$

46. By a numerical substitution show that $\dfrac{x^{n^2}}{x^n}$ does not equal x^2. Also that $\dfrac{x^{2ab}}{x^a}$ does not equal x^{2b}.

47. Write the square root of each of the following: $9x^{-4}$, $9x^{-3}$, $\frac{4}{9}x^{-\frac{1}{2}}$, $9x^{-\frac{2}{3}}$, $16a^{-\frac{3}{5}}$, $16a^{-\frac{1}{3}}$.

48. Solve $x^{-\frac{3}{2}} = 27$.

SUG. Raise both sides to the power $(-\frac{2}{3})$.

Then $(x^{-\frac{3}{2}})^{-\frac{2}{3}} = (27)^{-\frac{2}{3}} = \dfrac{1}{27^{\frac{2}{3}}} = \frac{1}{9}$. $\quad\therefore x = \dfrac{1}{9}$.

Find the value of x in each of the following:

49. $x^{\frac{1}{2}} = 2$.

50. $x^{\frac{3}{2}} = -27$.

51. $x^{-\frac{1}{2}} = 3$.

52. $x^{-\frac{2}{3}} = 4$.

53. $x^{-\frac{3}{5}} = -\frac{1}{8}$.

54. $x^{-\frac{5}{6}} = 1$.

55. $x^{-n} = 2$.

56. $x^{-\frac{1}{n}} = -3$.

57. $x^{-\frac{3}{4}} = -\frac{1}{27}$.

58. Make up and work an example similar to Ex. 11. To Ex. 28. Ex. 55.

59. Practice oral work with small fractions, as in Exercise 58 (p. 190).

186. Polynomials whose Terms contain Fractional or Negative Exponents.

Ex. 1. Multiply $x + 3x^{\frac{2}{3}} - 2x^{\frac{1}{3}}$ by $3 - 2x^{-\frac{1}{3}} + 4x^{-\frac{2}{3}}$.

$$\begin{array}{lllll}
x + 3x^{\frac{2}{3}} & - 2x^{\frac{1}{3}} & & & \\
3 - 2x^{-\frac{1}{3}} & + 4x^{-\frac{2}{3}} & & & \\
\hline
3x + 9x^{\frac{2}{3}} & - 6x^{\frac{1}{3}} & & & \\
\quad - 2x^{\frac{2}{3}} & - 6x^{\frac{1}{3}} & + 4 & & \\
 & + 4x^{\frac{1}{3}} & + 12 & - 8x^{-\frac{1}{3}} & \\
\hline
3x + 7x^{\frac{2}{3}} & - 8x^{\frac{1}{3}} & + 16 & - 8x^{-\frac{1}{3}} & \textit{Product}
\end{array}$$

Ex. 2. Extract the square root of

$$9a^{-3} - \frac{30}{\sqrt{a^5}} + \frac{13}{a^2} + 20\sqrt{a^3} + 4a^{-1}$$

Writing the expression by use of exponents only

$$\begin{array}{l|l}
 & 9a^{-3} - 30a^{-\frac{5}{2}} + 13a^{-2} + 20a^{-\frac{3}{2}} + 4a^{-1}\,)\,\underline{3a^{-\frac{2}{3}} - 5a^{-1} - 2a^{-\frac{1}{2}}} \quad \textit{Root} \\
 & 9a^{-3} \\
\hline
6a^{-\frac{3}{2}} - 5a^{-1} & -30a^{-\frac{5}{2}} + 13a^{-2} \\
\qquad\quad - 5a^{-1} & -30a^{-\frac{5}{2}} + 25a^{-2} \\
\hline
6a^{-\frac{3}{2}} - 10a^{-1} \quad - 2a^{-\frac{1}{2}} & -12a^{-2} + 20a^{-\frac{3}{2}} + 4a^{-1} \\
 & \underline{-12a^{-2} + 20a^{-\frac{3}{2}} + 4a^{-1}}
\end{array}$$

EXERCISE 91

1. Arrange $x^2 + 5x^{-1} + x^{-2} + x$ in ascending order of powers of x.

Arrange $x^{-n} + 1 + x^{-\frac{n}{2}}$ in descending order.

Multiply:

2. $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1$ by $a^{\frac{1}{3}} + 1$.

3. $3x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + 3y^{\frac{2}{3}}$ by $3x^{\frac{1}{3}} + 2y^{\frac{1}{3}}$.

4. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 4$ by $2 + 3x^{-\frac{1}{3}}$.

5. $a^{-1} - a^{-\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{-1} + a^{-\frac{1}{2}}b^{\frac{1}{2}} + b$.

6. Multiply $x^2 - xy + 2y^2$ by $2x^{-2} + x^{-1}y^{-1} + y^{-2}$.

7. Multiply $2x^{\frac{1}{2}} - 3y^{-1} + x^{-\frac{1}{2}}y^{-2}$ by $2x^{-\frac{1}{3}}y + 3x^{-\frac{5}{6}}$.

First try to multiply the following expressions as they stand; afterward rearrange the terms and multiply:

8. $2x^{\frac{2}{3}} - 4 - 3x^{\frac{1}{3}} + x^{-\frac{1}{3}}$ by $3x^{\frac{4}{3}} - 2x^{\frac{2}{3}} + x$.

9. $2 + a^{\frac{2}{3}}x^{-\frac{3}{4}} + a^{-\frac{2}{3}}x^{\frac{3}{4}}$ by $2a^{-\frac{2}{3}}x^{\frac{3}{4}} - 4a^{-\frac{4}{3}}x^{\frac{3}{2}} + 2a^{-2}x^{\frac{9}{4}}$.

10. $2\sqrt{x} + 3x^{\frac{1}{6}}\sqrt[3]{y} + \dfrac{4\sqrt[3]{y^2}}{\sqrt[6]{x}}$ by $\dfrac{3}{\sqrt{x}} - \dfrac{3}{\sqrt[6]{xy^2}} + \dfrac{2\sqrt[6]{x}}{\sqrt[3]{y^2}}$.

Divide:

11. $5x + 2x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$ by $x^{\frac{1}{3}} + 1$.

12. $8x^{-2} + \dfrac{6}{xy} + 3y^{-2} - 18xy^{-3} - 8x^2y^{-4}$ by $2x^{-1} + 3y^{-1} + 4xy^{-2}$.

13. $x^{-\frac{1}{2}} - x^{-\frac{1}{6}} + 5 - 2x^{\frac{1}{6}}$ by $1 + 2\sqrt[6]{x}$.

14. $\sqrt[3]{x} - \sqrt[3]{y}$ by $\sqrt{x} - \sqrt[9]{y}$.

15. $\sqrt{a} + \sqrt[4]{ab} + \sqrt{b}$ by $\sqrt[4]{a} + \sqrt[8]{ab} + \sqrt[4]{b}$.

16. $27a^2 - 30ay^{-1} + 3y^{-2}$ by $3a - 2a^{\frac{1}{2}}y^{-\frac{1}{2}} - y^{-1}$.

17. $x^{-\frac{4}{3}} + x^{-\frac{2}{3}}y^{-2} + y^{-4}$ by $x^{-1} + x^{-\frac{2}{3}}y^{-1} + x^{-\frac{1}{3}}y^{-2}$.

18. $\dfrac{x}{y} - x^{\frac{2}{3}}y^{-\frac{1}{2}} - 4\sqrt[3]{x} - \dfrac{8y}{\sqrt[3]{x}}$ by $\sqrt[3]{x} + 2y^{\frac{1}{2}}$.

19. $\dfrac{9}{a} - \dfrac{3\sqrt{x}}{\sqrt[4]{a^3}} + \dfrac{10x}{\sqrt{a}} + \dfrac{\sqrt{x^3}}{\sqrt[4]{a}} - x^2$ by $\dfrac{3}{\sqrt[4]{a}} + \sqrt{x}$.

20. $4\sqrt[3]{a^2} - 8\sqrt[3]{a} - 5 + \dfrac{10}{\sqrt[3]{a}} + \dfrac{3}{\sqrt[3]{a^2}}$ by $2a^{\frac{5}{12}} - \sqrt[12]{a} - \dfrac{3}{\sqrt[4]{a}}$.

Extract the square root of

21. $x^{\frac{2}{3}} - 4x^{\frac{5}{6}}y^{\frac{1}{2}} + 4xy$.

22. $9xy^{-2} + 12y^{-1} + 4x^{-1}$.

23. $a^{-1} - 4a^{-\frac{1}{2}}b^{\frac{1}{2}} + 10b - 12a^{\frac{1}{2}}b^{\frac{3}{2}} + 9ab^2$.

24. $x^{-\frac{3}{2}} + 8x^{-1} - 2x^{-\frac{3}{4}} + 16x^{-\frac{1}{2}} - 8x^{-\frac{1}{4}} + 1.$

25. $9x^{-3} - 30x^{-\frac{5}{2}}y + 13x^{-2}y^2 + 20x^{-\frac{3}{2}}y^3 + 4x^{-1}y^4.$

26. $25a^{\frac{4}{3}}b^{-3} - 10a^{\frac{2}{3}}b^{-\frac{3}{2}} - 49 + 10a^{-\frac{2}{3}}b^{\frac{3}{2}} + 25a^{-\frac{4}{3}}b^3.$

27. $\frac{9}{x^4} - \frac{18\sqrt{y}}{x^3} + \frac{15y}{x^2} - \frac{6\sqrt{y^3}}{x} + y^2.$

28. $9x^5 - \frac{24\sqrt{x^7}}{\sqrt[3]{y}} + \frac{4x^2}{\sqrt[3]{y^2}} + 16\sqrt{\frac{x}{y^2}} + \frac{4}{xy\sqrt[3]{y}}.$

29. $\frac{y^2}{4x} - \frac{y}{3x^{\frac{3}{4}}} + \frac{28}{9\sqrt{x}} - \frac{2}{y\sqrt[4]{x}} + \frac{9}{y^2}.$

30. Make up and work an example similar to Ex. 5. To Ex. 14. Ex. 24.

31. Practice the oral solution of simple equations as in Exercise 64 (p. 209).

EXERCISE 92

Review

1. Simplify $(\frac{4}{9})^{\frac{3}{2}} + 4^0 - 8^{\frac{2}{3}}$. Also $(\frac{4}{9})^{-\frac{3}{2}} \cdot 4^0 \cdot 8^{\frac{2}{3}}$.

2. Which is greater, $(2^3)^4$ or $2^3 \cdot 2^4$? How many times greater?

3. Which is greater, $(4^{-\frac{3}{2}})^6$ or $4^{-\frac{3}{2}} \cdot 4^6$? How many times greater?

4. Does $(x^a)^2$ equal x^{a^2}? Illustrate numerically.

5. Divide $9a^5 - 21a^3\sqrt{x} - \frac{25}{2}ax + 12a^{-1}x^{\frac{3}{2}}$ by $\frac{3}{2}a^3 - 4ax^{\frac{1}{2}}$.

Simplify:

6. $\frac{x^n(x^{n-1})^n}{x^{n+1} \cdot x^{n-1}}.$

7. $(x^{a-b})^{a+b} \div x^{-b^2}.$

8. $z^{a-b} \cdot z^{b+a} \div (z^2)^a.$

9. $(x^a)^{a+b} \cdot (x^b)^{a-b} \div (x^{a+2b})^a.$

10. Solve $3x^{\frac{2}{3}} = 32 - 5x^{\frac{2}{3}}$.

11. Simplify $\frac{\sqrt[3]{a}}{\sqrt[4]{b}}\left(\frac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \frac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}.$

12. Arrange and extract the square root of

$$x^{-2} + y^{\frac{4}{3}} + 2x^{-\frac{3}{2}}y^{\frac{1}{3}} - 2x^{-\frac{1}{2}}y - x^{-1}y^{\frac{2}{3}}.$$

13. Find the value of $(x^{\frac{3n}{2}} - y^{-\frac{3n}{2}}) \div (x^{\frac{n}{2}} - y^{-\frac{n}{2}})$.

14. Simplify $\frac{a}{b}\sqrt[3]{\frac{a^{-2}b^{0}}{ab^{-3}}}$.

15. Simplify $\frac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}}\sqrt[p]{a^{2pq-p^2}}$.

16. Simplify $\left[(a^n)^{n-\frac{1}{n}}\right]^{\frac{1}{n-1}}$.

Factor:

17. $x^{-2}-y^{\frac{1}{2}}$.

18. $x^{\frac{4}{3}}-5x^{\frac{2}{3}}+6$.

19. $x^{\frac{3}{5}}-y^{\frac{3}{5}}$.

20. Divide a^2-b^3 by $\sqrt[3]{a}-\sqrt{b}$.

21. Find the difference between the value of $(\frac{1}{2})^{-2}$ and that of $(-2)^{-2}$. Also between $(-2)^{-2}$ and -2^{-2}.

22. Does $2a^{-3}$ equal $\frac{1}{2a^3}$? Why? Give a numerical illustration.

23. Does $\frac{x^2+a^{-3}y}{5b}$ equal $\frac{x^2+y}{5a^3b}$? Why? Illustrate by giving a, b, x, and y convenient numerical values.

24. Does $a^{-\frac{2}{3}}+b^{-2}$ equal $\frac{1}{a^{\frac{2}{3}}+b^2}$? Explain as in Ex. 23.

25. Make up an example similar to each of the three preceding examples.

26. Show that $\frac{3x^4(x^2-a^2)^{\frac{1}{2}}-3x^2(x^2-a^2)^{\frac{3}{2}}}{x^6}$ reduces to $\frac{3a^2(x^2-a^2)^{\frac{1}{2}}}{x^4}$.

27. Reduce $(n-1)x^n(x^n+n)^{-\frac{1}{n}}+(x^n+n)^{\frac{n-1}{n}}$ to $\frac{n(x^n+1)}{(x^n+n)^{\frac{1}{2}}}$.

28. Show that $\frac{\frac{1}{3}(x+a)^{-\frac{2}{3}}-\frac{1}{3}(x-a)^{-\frac{2}{3}}}{(x+a)^{\frac{1}{3}}-(x-a)^{\frac{1}{3}}}=\frac{(x+a)^{\frac{1}{3}}+(x-a)^{\frac{1}{3}}}{3(x^2-a^2)^{\frac{2}{3}}}$.

29. Express 8^3 as a power of 2. Also $4^3\cdot 8^{\frac{2}{3}}\cdot 8^{-\frac{2}{3}}$, $4^n\cdot 4^{n+2}\cdot 8^{n-1}$.

30. Simplify $\frac{4^n(2^{n-1})^n}{2^{n+1}\cdot 4^2}$.

31. Expand $(x^{\frac{2}{3}}-4x^{-\frac{1}{2}})^3$.

32. Expand $(\sqrt{x}-2\sqrt[3]{x})^4$.

33. In the year 1910 the record time for 1 mile traveled on a bicycle was 1 min. 7 sec., which was $12\frac{1}{3}$ sec. more than twice the record for 1 mile traveled by an automobile. Find the latter record.

34. How many pounds of 18¢ coffee must be mixed with how many pounds of 30¢ coffee to make a mixture of 100 lb. worth 22¢ a pound?

35. Who first suggested the use of a fractional exponent and when? Who first showed that such exponents could be used according to mathematical laws? Who first used zero as an exponent and when?

EXERCISE 93

ORAL

Give the value of each of the following:

1. $4^{\frac{3}{2}}, 4^{-\frac{3}{2}}, 4^{-2}, 4^0, (4^{-2})^{-\frac{1}{2}}, (4^{-2})^{\frac{3}{2}}, \frac{1}{4^{-2}}$.

2. $(\frac{4}{9})^{\frac{3}{2}}, (\frac{4}{9})^0, (\frac{4}{9})^{-\frac{1}{2}}, 4^{\frac{1}{2}} + (\frac{4}{9})^{\frac{1}{2}}$.

3. $x^{a+b} \cdot x^{a-b} \cdot (x^{a+b})^{a-b}$.

4. $x^{2a+b} x^{2a-b} (x^{2a+b})^{2a-b}$.

5. $x^{\frac{n+1}{2}} x^{\frac{n-1}{2}}$.

6. $2^{n+1} \cdot 2^{n-1}$.

7. $2^{\frac{1}{2}} \cdot 2^{\frac{3}{2}}, 3^{\frac{3}{2}} \cdot 3^{-\frac{1}{2}}$.

8. $4^{\frac{1}{4}} \cdot 4^2, 4^{\frac{3}{2}} \cdot 4^0, 7x^0 + 5a^0$.

9. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

10. $(a^{\frac{3}{2}} + b^{\frac{3}{2}})(a^{\frac{3}{2}} - b^{\frac{3}{2}})$.

11. $(a^{\frac{3}{2}} - b^{\frac{3}{2}}) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

12. $(x^{-n} + y)(x^{-n} - y)$.

13. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$.

14. $(a^{-\frac{3}{2}} + b^{\frac{1}{2}})^2$.

15. $\frac{x^{-3} - y^{-3}}{x^{-1} - y^{-1}}$.

Factor:

16. $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.

17. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

18. $a^{-2} - b^{-2}$.

19. $x^{-4} - 9$.

20. Give the value of each of the following: $\frac{3^0}{5}, \frac{3}{5^0}, \frac{3^0}{5^0}, 3^0 \times 5, 3 \times 5^0, 3^0 \times 5^0, 3^0 + 5^0, 3^0 - 5$.

21. Give the value of $16^{\frac{3}{2}} \cdot 16^{\frac{1}{2}}$. Of $(16^{\frac{3}{2}})^{\frac{1}{2}}, 16^{-\frac{3}{2}} \cdot 16^{-\frac{1}{2}}, (16^{-\frac{3}{2}})^{-\frac{1}{2}}$.

22. Give the value of $\frac{x^{n^2}}{x^{2n}}$. Of $\frac{n^2}{2n}$. Of $x^{a+b} \cdot x^{2b-1} \cdot x^{1-a}$.

Give the value of x when

23. $x^{\frac{1}{2}} = 3$.

24. $x^{\frac{2}{3}} = 4$.

25. $x^{-\frac{1}{2}} = 2$.

26. $x^{-\frac{2}{3}} = 4$.

27. $x^{-2} = 3$.

28. $x^{-\frac{3}{4}} = 8$.

29. Give the square root of each of the following: $4x^{-\frac{2}{5}}$, $4x^{-\frac{3}{5}}$, $4x^{-3}$, $a^{-3}b^{\frac{1}{2}}$, $9x^5y^{-\frac{1}{2}}$, $\frac{1}{4}a^3b^{\frac{1}{2}}$.

30. State the value of $\$8^0$. Of $\$8^{-2}$. Of $\$8^{-\frac{2}{3}}$.

31. Express 5^0 as the quotient of two powers of 5, and hence find its value.

32. Express 4^{-2} as the quotient of two powers of 4, and hence find its value. Treat $4^{-\frac{3}{2}}$ in the same way. Also x^{-4}.

33. Give the value of each of the following: $(.3)^{-2}$, $1^{-2}\times 1^0\times 1^{-\frac{1}{2}}$, $(.01)^{\frac{1}{2}}$, $1^{-\frac{3}{2}} \div 4^0$.

34. Give the reciprocal of 2. Of $\frac{2}{3}$, $-\frac{3}{4}$, 4^{-2}, $8^{-\frac{2}{3}}$, $1^{-\frac{2}{5}}$.

CHAPTER XVII

RADICALS

187. Indicated Roots. The root of a quantity may be indicated in two ways:

(1) By the use of a fractional exponent; as $a^{\frac{1}{3}}$.

(2) By the use of a radical sign; as $\sqrt[3]{a}$.

For some purposes, one of these methods is better; for some, the other method.

Thus, when we have $a^{\frac{2}{3}} \times a^{\frac{1}{3}} \times a^{-2}$, where the quantities are alike except in their exponents, it is usually better to use fractional exponents to indicate roots; but if we have $5\sqrt{3} - 7\sqrt{27} + 8\sqrt{12}$, where exponents are alike, but coefficients and bases unlike, it is usually better to use the radical sign to indicate roots.

In the preceding chapter we considered exponents; we have now to investigate the properties of radicals.

188. A radical is a root of a quantity indicated by the use of the radical sign; as $\sqrt{x}$, $\sqrt[3]{27}$.

The **radicand** is the quantity under the radical sign.

In treating radicals, we deal only with principal roots (see Art. 172, p. 279), unless the contrary is stated.

189. Surds. An indicated root which may be exactly extracted is said to be *rational;* as $\sqrt[3]{27}$, since the cube root of 27 is 3.

A **surd** is an indicated root which cannot be exactly extracted; as $\sqrt{3}$, $\sqrt[3]{5}$.

190. The Coefficient of a radical is the number prefixed to the radical proper, to show how many times the radical is taken.

Thus, the coefficient of $5\sqrt{3}$ is 5; of $6a\sqrt[3]{x}$ is $6a$.

191. Entire Surds. If a surd has unity for its coefficient, it is said to be *entire*.

192. The Degree of a radical is the number of the indicated root.

Thus, $\sqrt[3]{x}$ is a radical of the third degree.

193. Similar Radicals are those which have the same quantity under the radical sign and the same index. (The coefficients and signs of the radicals may be unlike. Hence, similar radicals must be alike in two respects, and may be unlike in two other respects.)

Thus, $5\sqrt{3}$, $-4\sqrt{3}$ are similar radicals.

194. Fundamental Principle. Since a radical and a quantity affected by a fractional exponent differ only in form, in investigating the properties of radicals we may use the properties obtained for fractional exponents.

Thus, since

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$$

$$\therefore \sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

TRANSFORMATIONS OF RADICALS

195. Simplification of a Quantity under the Radical Sign.

Ex. 1. Simplify $\sqrt[3]{56}$.

$$\sqrt[3]{56} = \sqrt[3]{8 \times 7} = 2\sqrt[3]{7} \textit{ Ans.} \quad \text{(Art. 194)}$$

Ex. 2. Simplify $5\sqrt{18a^3b^2c^5}$.

$$5\sqrt{18a^3b^2c^5} = 5\sqrt{9a^2b^2c^4 \times 2ac} = 15abc^2\sqrt{2ac} \quad \textit{Ans.}$$

Hence, in general,

Separate the quantity under the radical sign into two factors, one of which is the greatest perfect power of the same degree as the radical;

Extract the required root of this factor, and multiply the coefficient of the radical by the result;

The other factor remains under the radical sign.

196. A Fraction under the Radical Sign. To simplify when the quantity under the radical sign is a fraction,

Multiply both numerator and denominator of the fraction by such a quantity as will make the denominator a perfect power of the same degree as the radical;

Proceed as in Art. 195.

Ex. 1. Simplify $\sqrt[3]{\frac{40}{9}}$.

$$\sqrt[3]{\tfrac{40}{9}} = \sqrt[3]{\tfrac{40}{9} \times \tfrac{3}{3}} = \sqrt[3]{\tfrac{120}{27}} = \sqrt[3]{\tfrac{8}{27} \times 15} = \tfrac{2}{3}\sqrt[3]{15} \quad \textit{Ans.}$$

Ex. 2. Simplify $\sqrt{\dfrac{5ax^2}{18b}}$.

$$\sqrt{\frac{5ax^2}{18b}} = \sqrt{\frac{5ax^2}{18b} \times \frac{2b}{2b}} = \sqrt{\frac{10abx^2}{36b^2}}$$

$$= \sqrt{\frac{x^2}{36b^2} \times 10ab} = \frac{x}{6b}\sqrt{10ab} \quad \textit{Ans.}$$

In studying radicals in examples involving letters, the work may often be checked by the substitution of numerical values.

197. Meaning of Simplification. By simplification radicals are reduced to their prime form, so that it is made easier to determine, for instance, whether a number of given radicals are similar or not.

Thus, it is difficult to determine whether $7\sqrt{18}$ and $-5\sqrt{72}$ are similar, but when the given radicals are put in the form $21\sqrt{2}$ and $-30\sqrt{2}$, it is easy to see that they are similar.

Again, the radicals $(a-1)\sqrt{\dfrac{a+1}{a-1}}$ and $(a+1)\sqrt{\dfrac{a-1}{a+1}}$, although unlike in their present form, may be reduced not only to similar radicals, but to the same expression, $\sqrt{a^2-1}$.

The pupil should show this reduction for himself.

Hence, a **radical in its simplest form** is one whose radicand is integral and contains no factor which is a power of the same degree with the radical.

EXERCISE 94

Express in the simplest form:

1. $\sqrt{12}$.
2. $\sqrt{27}$.
3. $-\sqrt{20}$.
4. $-3\sqrt{28}$.
5. $\frac{2}{3}\sqrt{45}$.
6. $\frac{4}{5}\sqrt{50}$.
7. $\sqrt[3]{48}$.
8. $\sqrt[3]{24}$.
9. $\sqrt[3]{54}$.
10. $\frac{1}{2}\sqrt[3]{72}$.
11. $-\frac{5}{3}\sqrt{108}$.
12. $\sqrt[4]{48}$.
13. $\sqrt[4]{128a^5x^2}$.
14. $\sqrt[3]{250a^2b^6}$.
15. $\sqrt{99a}$.
16. $2\sqrt{4a^5x^2}$.
17. $a\sqrt{8a^3x^2}$.
18. $\sqrt{200a^7}$.
19. $\sqrt{147x^5y^6}$.
20. $-2\sqrt{63x^{15}y^{10}}$.
21. $\sqrt[3]{-81a^3x^5}$.
22. $\sqrt{a^2(x-y)^3}$.
23. $\sqrt{49x^3(a+1)^5}$.
24. $10\sqrt{\dfrac{12a^3c^4n}{25x^4}}$.
25. $\frac{3}{4}\sqrt{\dfrac{112x^5z^{11}}{9a^2}}$.

Simplify:

26. $\sqrt{\frac{2}{3}}$.
27. $2\sqrt{\frac{5}{2}}$.
28. $3\sqrt{\frac{5}{6}}$.
29. $\sqrt{33\frac{1}{3}}$.
30. $\sqrt{\frac{25}{24}}$.
31. $\sqrt{\dfrac{63a}{8x^2}}$.
32. $4\sqrt{\dfrac{45c^3}{32a^2x}}$.
33. $\frac{3}{2}\sqrt{\dfrac{2x}{3a}}$.
34. $\dfrac{2a}{b}\sqrt{\dfrac{8b^2}{27a}}$.
35. $3\sqrt[3]{\frac{3}{4}}$.

36. $5a\sqrt[3]{\frac{8}{25}}$.

37. $\sqrt[2n]{\frac{5}{x^n}}$.

38. $-6\sqrt[3]{\frac{7x^4}{36a^2b^3}}$.

39. $\frac{8}{3}\sqrt[4]{\frac{1}{2}}$.

40. $z^2\sqrt{\frac{48a^2bc^3}{5x^2yz^3}}$.

41. $a\sqrt[5]{\frac{64x^2}{a^3}}$.

42. $\sqrt{\frac{12(x-y)}{5(x+y)}}$.

43. $x\sqrt{\frac{3}{(x+1)^3}}$.

44. $(a+b)^2\sqrt[3]{\frac{-a^4}{(a+b)^5}}$.

45. $4xy\sqrt{\frac{147a^3b^4c}{320xy^3}}$.

46. Given $\sqrt{3} = 1.73205+$, find in the shortest way the value of $\sqrt{12}$. Of $\sqrt{27}$. $\sqrt{75}$. $\sqrt{243}$.

47. Using $\sqrt{5} = 2.23607+$, make up and work an example similar to Ex. 46.

48. Make up and work an example similar to Ex. 5. To Ex. 23. Ex. 31.

EXERCISE 95

Oral

Reduce by inspection:

1. $\sqrt{8}$.

2. $\sqrt{a^3}$.

3. $\sqrt[3]{a^8x^4}$.

4. $\sqrt{\frac{8}{9}}$.

5. $\sqrt{\frac{1}{2}}$.

6. $\sqrt{\frac{1}{3}}$.

7. $\sqrt{\frac{1}{a}}$.

8. $\sqrt[3]{\frac{1}{4}}$.

9. $\sqrt{\frac{4}{5}}$.

10. $3\sqrt{\frac{2}{3}}$.

11. $2\sqrt{\frac{3}{2}}$.

12 $\sqrt{\frac{a}{b}}$.

13. $\sqrt{\frac{5}{12}}$.

14. $2\sqrt{4\frac{1}{2}}$.

15. $\sqrt{\frac{4R^2}{3}}$.

16. $\sqrt{\frac{3R^2}{4}}$.

17. How many examples in Exercise 94 (p. 308) can you work at sight?

198. I. Making Entire Surds. It is sometimes desirable to introduce the coefficient of a radical under the radical sign. This may be done by reversing the process of Art. 195.

Ex. 1. Express $3\sqrt[3]{5}$ as an entire surd.

$$3\sqrt[3]{5} = \sqrt[3]{3^3 \times 5} = \sqrt[3]{135} \quad \textit{Ans.}$$

Ex. 2. $-2\sqrt[5]{3} = -\sqrt[5]{96} = \sqrt[5]{-96}$ *Ans.*

Thus, it is difficult to determine whether $7\sqrt{18}$ and $-5\sqrt{72}$ are similar, but when the given radicals are put in the form $21\sqrt{2}$ and $-30\sqrt{2}$, it is easy to see that they are similar.

Again, the radicals $(a-1)\sqrt{\frac{a+1}{a-1}}$ and $(a+1)\sqrt{\frac{a-1}{a+1}}$, although unlike in their present form, may be reduced not only to similar radicals, but to the same expression, $\sqrt{a^2-1}$.

The pupil should show this reduction for himself.

Hence, a **radical in its simplest form** is one whose radicand is integral and contains no factor which is a power of the same degree with the radical.

EXERCISE 94

Express in the simplest form:

1. $\sqrt{12}$.
2. $\sqrt{27}$.
3. $-\sqrt{20}$.
4. $-3\sqrt{28}$.
5. $\frac{2}{3}\sqrt{45}$.
6. $\frac{4}{5}\sqrt{50}$.
7. $\sqrt[3]{48}$.
8. $\sqrt[3]{24}$.
9. $\sqrt[3]{54}$.
10. $\frac{1}{2}\sqrt[3]{72}$.
11. $-\frac{5}{3}\sqrt{108}$.
12. $\sqrt[4]{48}$.
13. $\sqrt[4]{128a^5x^2}$.
14. $\sqrt[3]{250a^2b^6}$.
15. $\sqrt{99a}$.
16. $2\sqrt{4a^5x^2}$.
17. $a\sqrt{8a^3x^2}$.
18. $\sqrt{200a^7}$.
19. $\sqrt{147x^5y^6}$.
20. $-2\sqrt{63x^{15}y^{10}}$.
21. $\sqrt[3]{-81a^3x^5}$.
22. $\sqrt{a^2(x-y)^3}$.
23. $\sqrt{49x^3(a+1)^5}$.
24. $10\sqrt{\frac{12a^3c^4n}{25x^4}}$.
25. $\frac{3}{4}\sqrt{\frac{112x^5z^{11}}{9a^2}}$.

Simplify:

26. $\sqrt{\frac{2}{3}}$.
27. $2\sqrt{\frac{5}{2}}$.
28. $3\sqrt{\frac{5}{6}}$.
29. $\sqrt{33\frac{1}{3}}$.
30. $\sqrt{\frac{25}{24}}$.
31. $\sqrt{\frac{63a}{8x^2}}$.
32. $4\sqrt{\frac{45c^3}{32a^2x}}$.
33. $\frac{3}{2}\sqrt{\frac{2x}{3a}}$.
34. $\frac{2a}{b}\sqrt{\frac{8b^2}{27a}}$.
35. $3\sqrt[3]{\frac{3}{4}}$.

36. $5a\sqrt[3]{\frac{8}{25}}$.

37. $\sqrt[2n]{\dfrac{5}{x^n}}$.

38. $-6\sqrt[3]{\dfrac{7x^4}{36a^2b^3}}$.

39. $\frac{8}{3}\sqrt[4]{\frac{1}{2}}$.

40. $z^2\sqrt{\dfrac{48a^2bc^3}{5x^2yz^3}}$.

41. $a\sqrt[5]{\dfrac{64x^2}{a^3}}$.

42. $\sqrt{\dfrac{12(x-y)}{5(x+y)}}$.

43. $x\sqrt{\dfrac{3}{(x+1)^3}}$.

44. $(a+b)^2\sqrt[3]{\dfrac{-a^4}{(a+b)^5}}$.

45. $4xy\sqrt{\dfrac{147a^3b^4c}{320xy^3}}$.

46. Given $\sqrt{3} = 1.73205+$, find in the shortest way the value of $\sqrt{12}$. Of $\sqrt{27}$. $\sqrt{75}$. $\sqrt{243}$.

47. Using $\sqrt{5} = 2.23607+$, make up and work an example similar to Ex. 46.

48. Make up and work an example similar to Ex. 5. To Ex. 23. Ex. 31.

EXERCISE 95

ORAL

Reduce by inspection:

1 $\sqrt{8}$.

2. $\sqrt{a^3}$.

3. $\sqrt[3]{a^8x^4}$.

4. $\sqrt{\frac{8}{9}}$.

5. $\sqrt{\frac{1}{2}}$.

6. $\sqrt{\frac{1}{3}}$.

7. $\sqrt{\dfrac{1}{a}}$.

8. $\sqrt[3]{\frac{1}{4}}$.

9. $\sqrt{\frac{4}{5}}$.

10. $3\sqrt{\frac{2}{3}}$.

11. $2\sqrt{\frac{3}{2}}$.

12 $\sqrt{\dfrac{a}{b}}$.

13. $\sqrt{\frac{5}{12}}$.

14. $2\sqrt{4\frac{1}{2}}$.

15. $\sqrt{\dfrac{4R^2}{3}}$.

16. $\sqrt{\dfrac{3R^2}{4}}$.

17. How many examples in Exercise 94 (p. 308) can you work at sight?

198. I. Making Entire Surds. It is sometimes desirable to introduce the coefficient of a radical under the radical sign. This may be done by reversing the process of Art. 195.

Ex. 1. Express $3\sqrt[3]{5}$ as an entire surd.

$$3\sqrt[3]{5} = \sqrt[3]{3^3 \times 5} = \sqrt[3]{135} \quad \textit{Ans.}$$

Ex. 2. $-2\sqrt[5]{3} = -\sqrt[5]{96} = \sqrt[5]{-96}$ *Ans.*

EXERCISE 96

Express as entire surds:

1. $2\sqrt{3}$.

2. $3\sqrt{5}$.

3. $6\sqrt[3]{2}$.

4. $-2\sqrt{5}$.

5. $-3\sqrt[3]{2}$

6. $-2\sqrt[3]{-2}$.

7. $2\sqrt[4]{3}$.

8. $2m\sqrt{3m}$.

9. $m^2\sqrt[3]{4n^2}$.

10. $\frac{1}{3}\sqrt{6}$.

11. $\frac{2}{5}\sqrt{10}$.

12. $\frac{3m}{4n}\sqrt[3]{\frac{2n}{9m^2}}$.

13. $2x\sqrt{\frac{3x-y}{x}}$.

14. $3a\sqrt[3]{\frac{a+2}{6a^2}}$.

15. $(x-1)\sqrt{2x}$.

16. $(a+2y)\sqrt{\frac{a-2y}{a+2y}}$.

17. $\frac{a-b}{a+b}\sqrt{a^2-b^2}$.

18. $(1-x)\sqrt{\frac{1}{x-1}}$.

19. Make up and work an example similar to Ex. 5.

20. To Ex. 7. Ex. 16.

21. Work again Exercise 76 (p. 249), or similar examples suggested by the teacher or pupils.

22. How many of the examples in this Exercise can you work at sight?

199. II. Simplification of Indices. If the exponent of the quantity under the radical sign and the index of the radical sign have a common factor, this factor may be canceled and the radical thereby simplified.

Ex. 1. $\sqrt[6]{a^2} = a^{\frac{2}{6}} = a^{\frac{1}{3}} = \sqrt[3]{a}$ *Ans.*

Ex. 2. $\sqrt[6]{125} = \sqrt[6]{5^3} = \sqrt{5}$ *Ans.*

EXERCISE 97

Simplify the indices of the following:

1. $\sqrt[4]{a^2}$.

2. $\sqrt[6]{a^3}$.

3. $\sqrt[4]{a^2y^4}$.

4. $\sqrt[4]{49}$.

5. $\sqrt[6]{27a^3}$.

6. $\sqrt[4]{100a^2x^8}$.

7. $\sqrt[9]{8a^3b^6c^9}$. **9.** $\sqrt[10]{32a^5x^{10}y^{15}}$. **11.** $\sqrt[5n]{x^ny^{2n}z^{3n}}$.

8. $\sqrt[12]{81a^4x^8}$. **10.** $\sqrt[2n]{9x^2y^4z^{10}}$. **12.** $6\sqrt[4]{2\frac{7}{9}}$.

13. $\sqrt[4]{a^2 - 4ab^2 + 4b^2}$. **14.** $\sqrt[6]{8(a-2b)^3}$.

15. Make up and work an example similar to Ex. 4.

16. To Ex. 7. Ex. 13.

200. III. Reducing Radicals to the Same Index. Radicals of different degrees may be reduced to equivalent radicals of the same degree.

Ex. 1. Reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to equivalent radicals having the same index.

$$\left.\begin{aligned}\sqrt{2} &= 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}\\ \sqrt[3]{5} &= 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}\end{aligned}\right\} \textit{Ans.}$$

Ex. 2. Arrange in ascending order of magnitude $\sqrt[4]{5}$, $\sqrt[3]{3}$, and $\sqrt{2}$.

We obtain $\sqrt[12]{125}$, $\sqrt[12]{81}$, $\sqrt[12]{64}$

hence, the ascending order of magnitude is, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$ *Ans.*

EXERCISE 98

Reduce to equivalent radicals of the same (lowest) degree:

1. $\sqrt{7}$ and $\sqrt[3]{11}$. **7.** $\sqrt[3]{2}$, $\sqrt[9]{9}$, $\sqrt[6]{5}$.

2. $\sqrt[3]{5}$ and $\sqrt[4]{3}$. **8.** $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt[4]{a^3}$.

3. $\sqrt[4]{3}$ and $\sqrt[6]{5}$. **9.** $\sqrt[6]{3a}$, $\sqrt[4]{2b}$, $\sqrt[6]{5c}$.

4. $\sqrt{\frac{1}{2}}$ and $\sqrt[3]{\frac{2}{3}}$. **10.** $\sqrt[4]{x+y}$ and $\sqrt[5]{x-y}$.

5. $\sqrt[4]{100}$ and $\sqrt[3]{25}$. **11.** $\sqrt[n]{x^m}$ and $\sqrt[m]{x^n}$.

6. $\sqrt{6}$ and $\sqrt[6]{200}$. **12.** $\sqrt[x]{c^y}$, $\sqrt[y]{c^z}$, $\sqrt[z]{c^x}$.

Show which is greater:

13. $\sqrt{3}$ or $\sqrt[3]{4}$. **15.** $2\sqrt{5}$ or $3\sqrt[3]{11}$.

14. $\sqrt[3]{15}$ or $\sqrt{6}$. **16.** $\sqrt[3]{23}$ or $2\sqrt{2}$.

17. $\sqrt[4]{10}$ or $2\sqrt[6]{\frac{1}{2}}$.

18. $\sqrt{2\frac{3}{5}}$ or $\sqrt[3]{4\frac{1}{4}}$.

19. $\frac{1}{2}\sqrt{7}$ or $\sqrt[3]{3}$.

20. $\sqrt{\frac{3}{5}}$ or $\sqrt[3]{\frac{7}{15}}$.

Show which is the greatest:

21. $\sqrt{3}$, $\sqrt[3]{5}$, or $\frac{1}{2}\sqrt[3]{40\frac{4}{5}}$.

22. $3\sqrt[3]{4\frac{1}{2}}$, $2\sqrt{6}$, or $2\sqrt[3]{14\frac{1}{2}}$.

23. Make up and work an example similar to Ex. 1.

24. To Ex. 7. Ex. 16.

25. How many examples in Exercise 83 (p. 273) can you now work at sight?

OPERATIONS WITH RADICALS

I. ADDITION AND SUBTRACTION OF RADICALS

201. The Addition of Similar Radicals is performed like the addition of similar terms, by *taking the algebraic sum of the coefficients of the terms.*

The addition of **dissimilar radicals** *can only be indicated.*

Ex. 1. Add $\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18}$.

$$\begin{aligned}\sqrt{128} - 2\sqrt{50} + \sqrt{72} - \sqrt{18} &= 8\sqrt{2} - 10\sqrt{2} + 6\sqrt{2} - 3\sqrt{2}\\ &= \sqrt{2}\ \textit{Sum}\end{aligned}$$

Ex. 2. $2\sqrt{\frac{5}{3}} + \frac{1}{6}\sqrt{60} + \sqrt{15} + \sqrt{\frac{3}{5}}$.

$$\begin{aligned}&= \tfrac{2}{3}\sqrt{15} + \tfrac{1}{3}\sqrt{15} + \sqrt{15} + \tfrac{1}{5}\sqrt{15}\\ &= \tfrac{11}{5}\sqrt{15}\ \textit{Sum}\end{aligned}$$

Ex. 3. $\sqrt[3]{128} + 2\sqrt[3]{\frac{1}{4}} - 3\sqrt[3]{81}$.

$$\begin{aligned}&= 4\sqrt[3]{2} + \sqrt[3]{2} - 9\sqrt[3]{3}\\ &= 5\sqrt[3]{2} - 9\sqrt[3]{3}\ \textit{Sum}\end{aligned}$$

EXERCISE 99

Collect:

1. $\sqrt{18} + \sqrt{8}$.

2. $\sqrt{50} - \sqrt{32}$.

3. $2\sqrt{27} + \sqrt{75}$.

4. $\sqrt{5} + \sqrt{20} + \sqrt{45}$.

5. $4\sqrt[3]{16} - 2\sqrt[3]{54}$.

6. $3\sqrt[3]{625} - 4\sqrt[3]{135}$.

7. $2\sqrt[3]{189} - \sqrt[3]{448}$.

8. $\sqrt[3]{24} + \sqrt[3]{81} - \sqrt[3]{375}$.

9. $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}$.

10. $2\sqrt{\frac{3}{4}} + \sqrt{48}$.

11. $a\sqrt{\frac{x}{a}} + x\sqrt{\frac{a}{x}}$.

12. $\sqrt[3]{\frac{5}{16}} + \frac{1}{4}\sqrt[3]{160}$.

13. $\frac{2}{3}\sqrt[3]{\frac{3}{4}} - \frac{3}{4}\sqrt[3]{\frac{2}{9}}$.

14. $2\sqrt{25b} + 3\sqrt{4b} - 2\sqrt{36b}$.

15. $3\sqrt[3]{2c} + 3\sqrt[3]{54c} - \sqrt[3]{2000c}$.

16. $\sqrt{12ab^2} + b\sqrt{48a} - 6\sqrt{3ab^2}$.

17. $2\sqrt{a^2bc^2} - 3a\sqrt{16bc^2} + 5c\sqrt{9a^2b}$.

18. $b\sqrt[3]{2a} + \sqrt[3]{250ab^3} - 2b\sqrt[3]{432a}$.

19. $\sqrt{2} + \sqrt{18} - \sqrt{50} + \sqrt{162}$.

20. $\sqrt{75} - 4\sqrt{243} + 2\sqrt{108}$.

21. $6\sqrt{\frac{2}{3}} - 5\sqrt{24} + 12\sqrt{\frac{3}{2}}$.

22. $5\sqrt{\frac{3}{5}} - 12\sqrt{\frac{5}{3}} + 6\sqrt{60} - 30\sqrt{\frac{1}{15}}$.

23. $3\sqrt{5} - 10\sqrt{\frac{1}{5}} + 2\sqrt{45} - 5\sqrt{\frac{49}{5}}$.

24. $\sqrt{27} - \sqrt{18} + \sqrt{300} - \sqrt{162} + 6\sqrt{2} - 7\sqrt{3}$.

25. $2\sqrt{63} - 3\sqrt{\frac{1}{5}} - \sqrt{\frac{9}{7}} + \frac{1}{5}\sqrt{45} - \frac{4}{7}\sqrt{7}$.

26. $\sqrt{243} + \sqrt{48} - \sqrt{768} + 9\sqrt{\frac{}{3}} + \sqrt{75} - 3\sqrt{33\frac{1}{3}}$.

27. $21\sqrt{\frac{2}{3}} - 5\sqrt{\frac{4}{5}} + 6\sqrt{4\frac{1}{6}} - 10\sqrt{3\frac{1}{5}} + \frac{40}{3}\sqrt{11\frac{1}{4}}$.

28. $5a\sqrt{12ab^2} - 3b\sqrt{27a^3} + 2\sqrt{300a^3b^2} - 40ab\sqrt{\frac{3}{4}a}$.

29. $3\sqrt[3]{16} - 3\sqrt{12} + 2\sqrt[3]{54} - 5\sqrt{27}$.

30. $3\sqrt[3]{54} - 2\sqrt{18} + 5\sqrt[3]{\frac{1}{4}} + 5\sqrt{\frac{1}{2}}$.

31. $3\sqrt{a^2x + 2abx + b^2x} + 2\sqrt{(a-b)^2x}$.

32. $8\sqrt{a(x+2y)^2} - 3\sqrt{ax^2 + 4axy + 4ay^2}$.

33. How many more symbols are used in Ex. 25 as it is given than in your answer?

34. Compute the value of $\sqrt{128}-2\sqrt{50}+\sqrt{72}$ by extracting the square root for each radical and combining results. Now obtain the value of the whole expression by first reducing each radical to its simplest form and then collecting. Compare the labor in the two processes.

35. Make up and work an example similar to Ex. 34.

36. Name some of the advantages in being able to reduce radicals to their simplest form.

37. Compute to three decimal places the value of

$\sqrt{20}-\sqrt{\frac{4}{5}}+2\sqrt{45}-\sqrt{\frac{1}{5}}$ in the shortest way.

38. Make up and work an example similar to Ex. 13.

39. To Ex. 17. Ex. 29.

40. How many of the examples in this Exercise can you work at sight?

II. MULTIPLICATION OF RADICALS

202. Multiplication of Monomials.

Since

$$\begin{aligned} a\sqrt[n]{b}\times c\sqrt[n]{d} &= ac\sqrt[n]{b}\sqrt[n]{d} \\ &= ac\sqrt[n]{bd} \end{aligned}$$

we have the general rule,

Reduce the radicals if necessary to the same index;
Multiply the coefficients together for a new coefficient;
Multiply the quantities under the radical sign together for a new quantity under the radical sign;
Simplify the result.

Ex. 1. Multiply $5\sqrt{6}$ by $2\sqrt{3}$.

$$5\sqrt{6}\times 2\sqrt{3} = 10\sqrt{18} = 30\sqrt{2} \textit{ Product}$$

Ex. 2. $5\sqrt{2} \times 2\sqrt[3]{3} = 5\sqrt[6]{8} \times 2\sqrt[6]{9}$

$= 10\sqrt[6]{72}$ *Product*

203. Multiplication of Polynomials.

Ex. Multiply $3\sqrt{2} + 5\sqrt{3}$ by $3\sqrt{2} - \sqrt{3}$.

$$\begin{array}{r}
3\sqrt{2} + 5\sqrt{3} \\
3\sqrt{2} - \sqrt{3} \\
\hline
18 + 15\sqrt{6} \\
- 3\sqrt{6} - 15 \\
\hline
3 + 12\sqrt{6}
\end{array} \quad \textit{Product}$$

EXERCISE 100

Multiply:

1. $3\sqrt{5}$ by $\sqrt{15}$.
2. $2\sqrt[3]{4}$ by $3\sqrt[3]{6}$.
3. $2\sqrt{15}$ by $3\sqrt{35}$.
4. $\frac{2}{3}\sqrt{28}$ by $\frac{3}{4}\sqrt{35}$.
5. $\frac{1}{3}\sqrt{\frac{5}{6}}$ by $\frac{2}{5}\sqrt{\frac{15}{32}}$.
6. $4\sqrt[3]{\frac{2}{3}}$ by $\frac{1}{2}\sqrt[3]{\frac{4}{3}}$.
7. $\sqrt{2}$ by $\sqrt[3]{3}$.
8. $\sqrt[3]{2ax^2}$ by $\sqrt{6x}$.
9. $\sqrt[3]{9}$ by $\sqrt{6}$.
10. $\sqrt[4]{6}$ by $\sqrt[6]{4}$.
11. $\sqrt{\frac{2}{3}}$, $\sqrt{\frac{3}{5}}$, and $\sqrt{2\frac{1}{2}}$.
12. $\sqrt{\dfrac{2ab^3}{3x^2y}}$ by $\sqrt[3]{\dfrac{9x^4y}{4ab^4}}$.
13. $\sqrt{3} - \sqrt{6} + 2\sqrt{10}$ by $2\sqrt{2}$.
14. $4\sqrt{6} - 3\sqrt{3} + 3\sqrt{2}$ by $2\sqrt{6}$.
15. $10\sqrt{\frac{5}{4}} - 5\sqrt{\frac{9}{2}} + 14\sqrt{\frac{45}{56}}$ by $\frac{2}{5}\sqrt{\frac{8}{15}}$.
16. $3 + \sqrt{2}$ by $2 - 2\sqrt{2}$.
17. $2\sqrt{3} - 3\sqrt{2}$ by $4\sqrt{3} + 5\sqrt{2}$.
18. $\sqrt{7 + \sqrt{5}}$ by $\sqrt{7 - \sqrt{5}}$.
19. $\sqrt{a + \sqrt{x}}$ by $\sqrt{a - \sqrt{x}}$.

Multiply:

20. $4\sqrt{2} - 3\sqrt{3}$ by $3\sqrt{2} + 4\sqrt{3}$.

21. $\sqrt{2} + \sqrt{3} - \sqrt{5}$ by $\sqrt{2} - \sqrt{3} + \sqrt{5}$.

22. $2\sqrt{3} - 3\sqrt{6} - 4\sqrt{15}$ by $2\sqrt{3} + 3\sqrt{6} + 4\sqrt{15}$.

23. $3\sqrt{30} + 2\sqrt{5} - 3\sqrt{6}$ by $2\sqrt{5} + 3\sqrt{6}$.

24. $\frac{1}{2}\sqrt{8} + \sqrt{32} - \sqrt{48}$ by $3\sqrt{8} - \frac{1}{4}\sqrt{32} + 2\sqrt{12}$.

25. $12\sqrt{\frac{1}{6}} - 4\sqrt{\frac{1}{2}} + \frac{1}{3}\sqrt{216}$ by $6\sqrt{\frac{2}{3}} - 2\sqrt{\frac{3}{2}} + 3\sqrt{6}$.

26. $\sqrt{2x} + \sqrt{x - 1}$ by $\sqrt{3x}$.

27. $\sqrt{3x} + \sqrt{x + 1}$ by $\sqrt{x + 1}$.

28. $\sqrt{x - 1} - 3\sqrt{x + 1}$ by $2\sqrt{x + 1}$.

29. $-\frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2}$ by $-\frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}$.

30. $a - \sqrt{a - x} + \sqrt{a}$ by $\sqrt{a - x} + \sqrt{a}$.

31. $3\sqrt{2x} - 5\sqrt{x - 1}$ by $3\sqrt{2x} + 5\sqrt{x - 1}$.

32. $\sqrt{a + x} - \sqrt{a - x}$ by $\sqrt{a + x} + \sqrt{a - x}$.

33. $(2\sqrt{2} + \sqrt{3})(3\sqrt{2} - \sqrt{3})(3\sqrt{3} - \sqrt{2})$.

34. $(2\sqrt{x} + 2 + 3\sqrt{2})(6x - 5\sqrt{2x + 4})(3\sqrt{x+2} - 2\sqrt{2})$.

35. In the year 1910 the greatest mountain height climbed by man was 24,853 ft., which was 3441 ft. less than twice the height of Pike's Peak. Find the height of Pike's Peak.

36. If 5 mi. exceed 8 kilometers by 153 ft. 4 in., how many yards in a kilometer?

37. Make up and work an example similar to Ex. 4.

38. To Ex. 9.

39. How many examples in Exercise 85 (p. 280) can you now work at sight?

III. Division of Radicals

204. Division of Radicals. Reversing the process for multiplication, we have the rule,

If necessary, reduce the radicals to the same index;

Find the quotient of the coefficients for a new coefficient, and the quotient of the quantities under the radical signs for a new quantity under the radical;

Simplify the result.

Ex. 1. Divide $6\sqrt{8}$ by $3\sqrt{6}$.

$$\frac{6\sqrt{8}}{3\sqrt{6}} = 2\sqrt{\tfrac{4}{3}} = 2\sqrt{\tfrac{12}{9}} = \tfrac{4}{3}\sqrt{3} \quad \textit{Quotient}$$

Ex. 2. Divide $6\sqrt[3]{3}$ by $2\sqrt{2}$.

$$\frac{6\sqrt[3]{3}}{2\sqrt{2}} = \frac{6\sqrt[6]{3^2}}{2\sqrt[6]{2^3}} = 3\sqrt[6]{\frac{3^2 \times 2^3}{2^3 \times 2^3}} = 3\sqrt[6]{\frac{72}{2^6}} = \frac{3}{2}\sqrt[6]{72} \quad \textit{Quotient}$$

EXERCISE 101

Divide:

1. $\sqrt{27}$ by $\sqrt{3}$.
2. $4\sqrt{12}$ by $2\sqrt{6}$.
3. $8\sqrt{125}$ by $10\sqrt{10}$.
4. $3\sqrt{405}$ by $9\sqrt{45}$.
5. $a^3\sqrt{ab^3}$ by $2a\sqrt{a^3b}$.
6. $4\sqrt{18}$ by $5\sqrt{32}$.
7. $\sqrt{\tfrac{15}{16}}$ by $\sqrt{\tfrac{5}{8}}$.
8. $4\sqrt{\tfrac{5}{6}}$ by $3\sqrt{\tfrac{20}{27}}$.
9. $5\sqrt{\tfrac{22}{35}}$ by $2\sqrt{\tfrac{7}{11}}$.
10. $2\sqrt[3]{3}$ by $3\sqrt{2}$.
11. $\sqrt{54}$ by $\sqrt[4]{36}$.
12. $\sqrt[3]{12}$ by $\sqrt{6}$.
13. $\sqrt{\tfrac{8}{15}}$ by $\sqrt[3]{\tfrac{16}{135}}$.
14. $\sqrt{6\tfrac{1}{2}}$ by $\sqrt[3]{3\tfrac{1}{4}}$.
15. $5\sqrt{35} - 7\sqrt{20}$ by $\sqrt{5}$.
16. $12\sqrt{7} - 60\sqrt{5}$ by $4\sqrt{3}$.
17. $6\sqrt{105} + 18\sqrt{40} - 45\sqrt{12}$ by $3\sqrt{15}$.

18. Divide $12\sqrt[3]{45}+30\sqrt[3]{20}+42\sqrt[3]{30}$ by $2\sqrt[3]{18}$.

19. Divide $10\sqrt[3]{18}-4\sqrt[3]{60}+5\sqrt[3]{100}$ by $3\sqrt[3]{30}$.

20. Divide $\sqrt{x^2-y^2}$ by $\sqrt{x-y}$.

21. Make up and work an example similar to Ex. 3.

22. To Ex. 12. Ex. 19.

23. Work again such examples on pp. 239–240 as the teacher may indicate.

205. Rationalizing a Monomial Denominator. If the denominator of a fraction is a surd, in order to make the denominator rational,

Multiply both numerator and denominator by such a number as will make the denominator rational.

$$\text{Ex.}\quad \frac{5}{\sqrt[3]{2}}=\frac{5}{\sqrt[3]{2}}\times\frac{\sqrt[3]{4}}{\sqrt[3]{4}}=\frac{5\sqrt[3]{4}}{\sqrt[3]{8}}=\frac{5}{2}\sqrt[3]{4}\ \textit{Ans.}$$

One object in rationalizing the denominator of a fraction is to diminish the labor of finding the approximate value of the fraction.

Thus, if we find the approximate numerical value of $\frac{5}{\sqrt{2}}$ directly, we must find the square root of 2, and divide 5 by the decimal which we obtain. On the other hand, if we find the value of the equivalent expression, $\frac{5}{2}\sqrt{2}$, we extract the square root of 2, multiply by 5, and divide by 2. In the latter process we therefore avoid the tedious long division, and diminish the labor of the process by nearly one half.

206. Rationalizing a Binomial or Trinomial Denominator. If the denominator of a fraction is a binomial containing radicals of the second degree only, since

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$$

Multiply both numerator and denominator by the denominator, with one of its signs changed;

For a trinomial denominator repeat the process.

Ex. 1. $$\frac{2\sqrt{5}+4\sqrt{3}}{3\sqrt{5}-\sqrt{3}}=\frac{2\sqrt{5}+4\sqrt{3}}{3\sqrt{5}-\sqrt{3}}\times\frac{3\sqrt{5}+\sqrt{3}}{3\sqrt{5}+\sqrt{3}}$$

$$=\frac{42+14\sqrt{15}}{45-3}=\frac{42+14\sqrt{15}}{42}=\frac{3+\sqrt{15}}{3}\quad \textit{Ans.}$$

Ex. 2. $$\frac{4}{1+\sqrt{3}+\sqrt{2}}=\frac{4}{1+\sqrt{3}+\sqrt{2}}\times\frac{1+\sqrt{3}-\sqrt{2}}{1+\sqrt{3}-\sqrt{2}}$$

$$=\frac{2(1+\sqrt{3}-\sqrt{2})}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}=2+\sqrt{2}-\sqrt{6}\quad \textit{Ans.}$$

EXERCISE 102

Reduce to equivalent fractions having rational denominators:

1. $\dfrac{1}{\sqrt{2}}$.

2. $\dfrac{\sqrt{2}}{2\sqrt{3}}$.

3. $\dfrac{2}{3\sqrt{5}}$.

4. $\dfrac{1-\sqrt{2}}{\sqrt{6}}$.

5. $\dfrac{2+\sqrt{5}}{2\sqrt{7}}$.

6. $\dfrac{3\sqrt{2}-\sqrt{3}}{2\sqrt{6}}$.

7. $\dfrac{\sqrt[3]{5}-1}{3\sqrt[3]{2}}$.

8. $\dfrac{2\sqrt[3]{9}-3\sqrt[3]{4}}{5\sqrt[3]{6}}$.

9. $\dfrac{3-\sqrt{2}}{3+\sqrt{2}}$.

10. $\dfrac{5+\sqrt{3}}{2-\sqrt{3}}$.

11. $\dfrac{3\sqrt{3}-2\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$.

12. $\dfrac{3\sqrt{6}-2\sqrt{3}}{4\sqrt{6}-3\sqrt{3}}$.

13. $\dfrac{2\sqrt{15}+3\sqrt{10}}{4\sqrt{3}+3\sqrt{2}}$.

14. $\dfrac{3\sqrt{a}-4\sqrt{b}}{2\sqrt{a}-3\sqrt{b}}$.

15. $\dfrac{\sqrt{x+1}+3}{\sqrt{x+1}+2}$.

16. $\dfrac{2\sqrt{2a-1}+3\sqrt{a}}{3\sqrt{2a-1}+2\sqrt{a}}.$

17. $\dfrac{2+\sqrt{6}-\sqrt{2}}{2-\sqrt{6}+\sqrt{2}}.$

18. $\dfrac{\sqrt{5}-\sqrt{6}+1}{\sqrt{6}+\sqrt{5}+1}.$

19. $\dfrac{2\sqrt{6}-3\sqrt{2}-\sqrt{42}}{2\sqrt{6}+3\sqrt{2}+\sqrt{42}}.$

20. $\dfrac{\sqrt{x^2-1}-\sqrt{x^2+1}}{\sqrt{x^2-1}+\sqrt{x^2+1}}.$

21. $\dfrac{\sqrt{a}+\sqrt{b}-\sqrt{a+b}}{\sqrt{a}-\sqrt{b}+\sqrt{a+b}}.$

Find the approximate numerical value of

22. $\dfrac{3}{\sqrt{2}}.$

23. $\dfrac{2\sqrt{5}}{3\sqrt{2}}.$

24. $\dfrac{12}{\sqrt{7}}.$

25. $\dfrac{1}{\sqrt{300}}.$

26. $\dfrac{3\sqrt{7}}{5\sqrt{5}}.$

27. $\dfrac{1+\sqrt{2}}{2-\sqrt{3}}.$

28. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}+\sqrt{3}}.$

29. $\dfrac{5\sqrt{7}-1}{\sqrt{7}+2}.$

30. $\dfrac{3\sqrt{3}-4}{4\sqrt{3}-5}.$

31. Make up and work an example similar to Ex. 2.

32. To Ex. 14. Ex. 23. Ex. 29.

33. How many examples in Exercise 45 (p. 155) can you now work at sight?

IV. Involution and Evolution of Radicals

207. The process of raising a radical to a power, or of extracting a required root of a radical, is usually performed most readily by the use of fractional exponents.

Ex. 1. Find the square of $3\sqrt[3]{x}$.

$$(3\sqrt[3]{x})^2=(3x^{\frac{1}{3}})^2=9x^{\frac{2}{3}}=9\sqrt[3]{x^2} \quad Ans.$$

Ex. 2. Extract the square root of $4a\sqrt{a^3b^5}$.

$$(4a\sqrt{a^3b^5})^{\frac{1}{2}}=(4a\cdot a^{\frac{3}{2}}b^{\frac{5}{2}})^{\frac{1}{2}}=4^{\frac{1}{2}}a^{\frac{1}{2}}a^{\frac{3}{4}}b^{\frac{5}{4}}$$
$$=2a^{\frac{5}{4}}b^{\frac{5}{4}}=2\sqrt[4]{a^5b^5}=2ab\sqrt[4]{ab} \quad Ans.$$

Ex. 3. Extract the cube root of $\sqrt{a^6b^9}$.

$$(\sqrt{a^6b^9})^{\frac{1}{3}} = (a^3b^{\frac{9}{2}})^{\frac{1}{3}} = ab^{\frac{3}{2}} = a\sqrt{b^3} = ab\sqrt{b} \quad \textit{Ans.}$$

This process might have been performed by extracting the cube root of a^6b^9 as it stands under the radical sign; thus,

$$\sqrt[3]{\sqrt{a^6b^9}} = \sqrt{a^2b^3} = ab\sqrt{b} \quad \textit{Ans.}$$

EXERCISE 103

Perform the operations indicated:

1. $(\sqrt{m})^4$.

2. $(\sqrt[3]{x^2})^6$.

3. $(\sqrt[3]{a^4x^5})^3$.

4. $(\sqrt[6]{36})^3$.

5. $(\sqrt{a^3c^5})^4$.

6. $(\sqrt[5]{x^6})^{15}$.

7. $(\sqrt[3]{2x^4})^9$.

8. $(\sqrt{\sqrt[3]{a^2}})^3$.

9. $(\sqrt[3]{3x^2})^6$.

10. $\sqrt[3]{64a^3\sqrt{8x^3}}$.

11. $(\sqrt[4]{\sqrt[3]{x^4y^8}})^6$.

12. $\sqrt{\sqrt[3]{4\sqrt[4]{64x^{12}}}}$.

13. Show how the computation of $(\sqrt{2})^{10}$ may be shortened by the use of $(\sqrt{2})^2 = 2$, and also of the fact that $2^5 = 32$.

14. Compute the value of each of the following in the shortest way: $(\sqrt{2})^8$, $(\sqrt{3})^6$, $(\sqrt{3})^8$, $(\sqrt{2})^9$, $(\sqrt{2})^{12}$, $(-\sqrt{3})^{10}$, $(-\sqrt{3})^9$, $(\sqrt[4]{3})^{20}$.

Expand:

15. $(\sqrt{3}+\sqrt{2})^4$. **16.** $(\sqrt{3}-\sqrt{2})^5$. **17.** $(2\sqrt{3}-\frac{1}{2}\sqrt{2})^6$.

18. Make up and work an example similar to Ex. 14.

19. To Ex. 15.

20. Practice oral work with small fractions as in Exercise 58 (p. 190).

V. SQUARE ROOT OF A BINOMIAL SURD

208. A quadratic surd is a surd of the second degree; as $\sqrt{3}$ and $\sqrt{ab}$.

A **binomial surd** is a binomial expression, at least one term of which contains a surd; as $\sqrt{2}+5\sqrt{3}$, or $a+\sqrt{b}$.

209. A. *The product of two dissimilar quadratic surds is a quadratic surd.*

Thus, $\sqrt{2}\times\sqrt{6}=\sqrt{12}=2\sqrt{3}$

or $\sqrt{ab}\times\sqrt{abc}=ab\sqrt{c}$

PROOF. If the surds are dissimilar, one of them must have under the radical sign a factor which the other does not contain. This factor must remain under the radical sign in the product.

210. B. *The sum or the difference of two dissimilar quadratic surds cannot equal a rational quantity.*

We use $x \pm y$ as a short way of writing $x+y$ and $x-y$.

PROOF. If $\sqrt{a}\pm\sqrt{b}$ can equal a rational quantity, c,

squaring, $a\pm 2\sqrt{ab}+b=c^2$

$$\pm 2\sqrt{ab}=c^2-a-b$$

But $\sqrt{ab}$ is a surd by Art. 209; hence we have a surd equal to a rational quantity, which is impossible.

211. C. ***If*** $\boldsymbol{a+\sqrt{b}=x+\sqrt{y}}$***, then*** $\boldsymbol{a=x, b=y}$.

PROOF. If $a+\sqrt{b}=x+\sqrt{y}$

transposing, $\sqrt{b}-\sqrt{y}=x-a$

If b does not equal y, we have the difference of two surds equal to a rational quantity, which is impossible; hence,

$$b=y \qquad a=x$$

In like manner, show that if

$$a-\sqrt{b}=x-\sqrt{y} \qquad \text{then } a=x \qquad b=y$$

212. D. *If* $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$, *then* $x+y=a$.

Squaring the given equals, $x+y+2\sqrt{xy}=a+\sqrt{b}$

Hence, $x+y=a$ (Art. 211)

In like manner, if $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$

it may be shown that $x+y=a$

Also, since $2\sqrt{xy}=\sqrt{b}$ (Art. 211)

$$x+y-2\sqrt{xy}=a-\sqrt{b}$$

and $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$

213. Extraction of the Square Root of a Binomial Surd.

Ex. Extract the square root of $5+2\sqrt{6}$.

Let $\sqrt{x}+\sqrt{y}=\sqrt{5+2\sqrt{6}}\;\therefore x+y=5$ (Art. 212)

Then, $\sqrt{x}-\sqrt{y}=\sqrt{5-2\sqrt{6}}$ (Art. 212)

Multiplying, $x-y=\sqrt{25-24}\;\therefore x-y=1$

$$\therefore\; x=3$$

$$y=2$$

$$\therefore\; \sqrt{x}+\sqrt{y}=\sqrt{3}+\sqrt{2}$$

$$\therefore \sqrt{5+2\sqrt{6}}=\sqrt{3}+\sqrt{2} \textit{ Ans.}$$

214. Finding the Square Root of a Binomial Surd by Inspection.

By actual multiplication we may find,

$$(\sqrt{2}+\sqrt{5})^2=2+2\sqrt{10}+5=7+2\sqrt{10}$$

In the square, $7+2\sqrt{10}$, 7 is the sum of 2 and 5, 10 is the product of 2 and 5. Hence, in extracting the square root of a binomial surd,

Transform the surd term so that its coefficient shall be 2;

Find two numbers such that their sum shall equal the rational term, and their product equal the quantity under the radical;

Extract the square root of each of these numbers, and connect the results by the proper sign.

Ex. Find the square root of $18 + 8\sqrt{5}$.

$$18 + 8\sqrt{5} = 18 + 2\sqrt{80}$$

The two numbers whose sum is 18 and product is 80 are 8 and 10.

$$\therefore \sqrt{18 + 8\sqrt{5}} = \sqrt{8} + \sqrt{10}$$
$$= 2\sqrt{2} + \sqrt{10} \textit{ Root}$$

EXERCISE 104

Find the square root of

1. $17 - 12\sqrt{2}$.
2. $23 + 4\sqrt{15}$.
3. $35 - 12\sqrt{6}$.
4. $9 - 6\sqrt{2}$.
5. $42 + 28\sqrt{2}$.
6. $73 - 12\sqrt{35}$.
7. $26 + 4\sqrt{30}$.
8. $77 - 24\sqrt{10}$.
9. $87 - 36\sqrt{5}$.
10. $14 + 3\sqrt{3}$.
11. $8 - \frac{16}{3}\sqrt{2}$.
12. $5\frac{1}{4} + 3\sqrt{3}$.
13. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$.
14. $2m + 2\sqrt{m^2 - n^2}$.
15. $10a^2 + 9 + 6a\sqrt{a^2 + 1}$.

Find the fourth root of

16. $28 - 16\sqrt{3}$.
17. $97 - 56\sqrt{3}$.
18. $193 - 132\sqrt{2}$.
19. $\frac{49}{9} + \frac{20}{9}\sqrt{6}$.

Find by inspection the square root of

20. $3 + 2\sqrt{2}$.
21. $9 - 2\sqrt{14}$.
22. $21 + 12\sqrt{3}$.
23. $23 - 6\sqrt{10}$.
24. $18 - 12\sqrt{2}$.
25. $7 + 4\sqrt{3}$.
26. Prove that $\sqrt{a} \pm \sqrt{b}$ cannot equal $\sqrt{c}$.
27. Prove that $\sqrt{a}$ cannot equal $b + \sqrt{c}$.
28. Make up and work an example similar to Ex. 2.

29. To Ex. 11.

30. Practice the oral solution of simple equations as in Exercise 64 (p. 209).

VI. SOLUTION OF EQUATIONS CONTAINING RADICALS

215. Simple Equations containing Radicals.

Ex. 1. Solve $\sqrt{x^2+7}-1=x$.

Transpose terms so that the radical shall be alone on one side of the equation.

$$\sqrt{x^2+7}=x+1$$

Squaring, $$x^2+7=x^2+2x+1$$

$$\therefore 2x=6$$

$$x=3 \quad \textit{Root}$$

CHECK. $$\sqrt{x^2+7}=\sqrt{9+7}=4$$

$$x+1=3+1=4$$

Observe that only the principal value of a radical is used in checking a result, as in the other processes in this chapter.

Ex. 2. Solve $\sqrt{x+3}+\sqrt{x}=5$.

Transpose terms so that one radical shall be alone on one side of the equation.

$$\sqrt{x+3}=5-\sqrt{x}$$

Squaring, $$x+3=25-10\sqrt{x}+x$$

$$\therefore 10\sqrt{x}=22$$

$$5\sqrt{x}=11$$

Squaring, $$25x=121$$

$$x=\tfrac{121}{25} \quad \textit{Root}$$

Let the pupil check the work.

In general,

Transpose the terms of the given equation so that a single radical shall form one member of the equation;

Raise both members of the equation to the power indicated by the index of this radical;

Repeat the process if necessary.

216. Fractional Equations containing Radicals.

Ex. 1. $\sqrt{x} - \sqrt{x-8} = \dfrac{2}{\sqrt{x-8}}.$

Multiply by $\sqrt{x-8}$, $\sqrt{x^2-8x} - x + 8 = 2$

$$\begin{aligned} \therefore \sqrt{x^2-8x} &= x - 6 \\ x^2 - 8x &= x^2 - 12x + 36 \\ 4x &= 36 \\ x &= 9 \text{ } \textit{Root} \end{aligned}$$

Ex. 2. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2} = \dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}.$

Clearing of fractions, $3x - 4\sqrt{x} - 39 = 3x - 11\sqrt{x} + 10$

$$\begin{aligned} 7\sqrt{x} &= 49 \\ \sqrt{x} &= 7 \\ x &= 49 \text{ } \textit{Root} \end{aligned}$$

EXERCISE 105

Solve the following equations:

1. $3 - \sqrt{x+1} = 0.$
2. $5 - \sqrt{2x} = 3.$
3. $1 - \sqrt{3x-5} = 0.$
4. $1 = \sqrt[3]{x} - 1.$
5. $\sqrt[3]{2x-1} + 1 = 4.$
6. $x - 1 - \sqrt{x^2+3} = 0.$
7. $2\sqrt{3x-5} - 3\sqrt{x+1} = 0.$
8. $3\sqrt{x} - 1 = \sqrt{x} + 1.$
9. $\sqrt{x+16} - 8 + \sqrt{x} = 0.$
10. $\sqrt{x-15} - 15 + \sqrt{x} = 0.$
11. $\frac{3}{2} + \sqrt{x} = \sqrt{\frac{15}{4} + x}.$
12. $\frac{5}{2} - \sqrt{2x} - \sqrt{2x + \frac{5}{4}} = 0.$
13. $\sqrt{x} + \sqrt{x+8} - 8 = 0.$
14. $\sqrt{4x+3} = 2\sqrt{x-1} + 1.$
15. $2\sqrt{x} - \sqrt{4x-22} = \sqrt{2}.$
16. $\sqrt{9x+35} = 7\sqrt{5} - 3\sqrt{x}.$
17. $\sqrt{13 + \sqrt{7 + \sqrt{3 + \sqrt{x}}}} = 4.$
18. $\sqrt{\sqrt{x}+3} - \sqrt{\sqrt{x}-3} - \sqrt{2\sqrt{x}} = 0.$

19. $\sqrt{25x-29}-\sqrt{4x-11}-3\sqrt{x}=0.$

20. $\sqrt{x}+\sqrt{4a+x}-2\sqrt{b+x}=0.$

21. $\sqrt{3+x}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}.$

22. $\sqrt{9+2x}=\dfrac{5}{\sqrt{9+2x}}+\sqrt{2x}.$

23. $3\sqrt{2x+1}-3\sqrt{2x-3}=\dfrac{4}{\sqrt{2x-3}}.$

24. $\dfrac{\sqrt{x}-3}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}-2}.$

25. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}.$

26. $\dfrac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=a.$

27. $\dfrac{\sqrt{ax+b}+\sqrt{ax}}{\sqrt{ax+b}-\sqrt{ax}}=1+\dfrac{1}{b}.$

28. $\dfrac{x-a}{\sqrt{x}+\sqrt{a}}=\dfrac{\sqrt{x}-\sqrt{a}}{3}+2\sqrt{a}.$

29. $\dfrac{\sqrt{9x+2}-3\sqrt{x}}{\sqrt{9x+2}+3\sqrt{x}}=4.$

30. $\sqrt[4]{2\frac{1}{2}+\sqrt{1\frac{1}{4}+\sqrt{\frac{1}{2}+\sqrt{x}}}}=\sqrt{2}.$

31. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

32. $\dfrac{a+3}{\sqrt{x}+2}-\dfrac{a-3}{\sqrt{x}-2}=\dfrac{2a}{x-4}.$

33. $\sqrt{\dfrac{a}{x}+1}-\sqrt{\dfrac{a}{x}-1}=1.$

34. $\dfrac{1}{x}+\dfrac{1}{5}=\sqrt{\dfrac{1}{25}+\sqrt{\dfrac{1}{a^2x^2}+\dfrac{1}{x^4}}}.$

35. $\sqrt{x^2 + 4x + 12} + \sqrt{x^2 - 12x - 20} = 8.$

36. $\sqrt{9x^2 + x + 5} - \sqrt{9x^2 + 7x + 6} - 1 = 0.$

37. The square root of a certain number plus the square root of the sum of the number and 7 equals 7. Find the number.

38. State Ex. 1. above as a problem concerning one unknown number. Similarly state Ex. 4. Ex. 19.

39. Practice oral work with exponents as in Exercise 93 (p. 303).

217. Extraneous Roots in Radical Equations. It may readily be shown that *squaring each of the two members of an equation does not necessarily produce an equivalent equation.*

Ex. $5 + \sqrt{x} = 2$

Hence, $\sqrt{x} = -3$

$x = 9$

Checking, $5 + \sqrt{9} = 2$, or $8 = 2$. But this is impossible; hence, 9 is not a root of the given equation.

Note that if the sign of the radical $\sqrt{x}$ is changed in the original equation, by solving the equation thus formed the result $x = 9$ is obtained; this answer can be proved.

EXERCISE 106

Solve the following equations and check each result. In each case where the root is impossible, change the original equation so as to make the result obtained a root.

1. $1 - \sqrt{x} = 3.$
2. $4 - \sqrt{x + 1} = 5.$
3. $3 + \sqrt{2x} = 5.$
4. $\sqrt{x + 7} = 2 - \sqrt{x - 5}.$
5. $\sqrt{7x + 18} = \sqrt{7x + 1} + 1.$
6. $\sqrt{4x + 9} + \sqrt{x} = \sqrt{x + 5}.$
7. $\sqrt{x + 9} - \sqrt{x + 2} - \sqrt{4x - 27} = 0.$

8. $\sqrt{x} - \sqrt{x+8} = 8.$

9. $\sqrt{x+8} - 8 + \sqrt{x} = 0.$

10. Practice the oral simplification of radicals as in Exercise 95 (p. 309).

EXERCISE 107

REVIEW OF RADICALS

Collect:

1. $\frac{2}{3}\sqrt{45} - 30\sqrt{\frac{1}{5}} + \frac{3}{2}\sqrt{20} - 6\sqrt{\frac{80}{9}} + \sqrt{500}.$

2. $4\sqrt{147} - 3\sqrt{75} - 6\sqrt{\frac{1}{3}} + 18\sqrt{\frac{1}{27}} - 24\sqrt{\frac{3}{4}}.$

3. $\frac{2a}{3b}\sqrt{\frac{45b^2x}{16a^4}} - \frac{10b}{a^2}\sqrt{\frac{a^2x}{5b^2}} + \frac{14x}{a^2b}\sqrt{\frac{5a^2b^2}{49x}} - \frac{5x}{2a}\sqrt{\frac{1}{20x}}.$

Multiply:

4. $2 + \sqrt{3} - \sqrt{5}$ by $2 + \sqrt{3} + \sqrt{5}.$

5. $\sqrt{a} + \sqrt{a+x} - \sqrt{x}$ by $\sqrt{a} - \sqrt{a+x} - \sqrt{x}.$

6. $\frac{1}{2}\sqrt{\frac{6}{5}}$ by $3\sqrt[3]{\frac{25}{6}}$. Also $2\sqrt[4]{\frac{28}{15}}$ by $\frac{1}{3}\sqrt{\frac{225}{56}}.$

Divide:

7. $6\sqrt{12} + 3\sqrt{8} - 6\sqrt{30} + 4\sqrt{15}$ by $2\sqrt{6}.$

8. $x^2 - x + 1$ by $x + \sqrt{x-1}.$

Rationalize the denominator of

9. $\frac{2\sqrt{3} - 3\sqrt{2}}{3 - 2\sqrt{6}}.$

10. $\frac{2\sqrt{15} + 8}{5 + \sqrt{15}} \div \frac{8\sqrt{3} - 6\sqrt{5}}{5\sqrt{3} - 3\sqrt{5}}.$

Find the numerical value of each of the following in the shortest way:

11. $\frac{5}{\sqrt{3}}.$

12. $\frac{2\sqrt{5} - 3\sqrt{2}}{3\sqrt{5} - 4\sqrt{2}}.$

Which is the greater:

13. $3\sqrt{3}$ or $2\sqrt{7}$?

14. $\sqrt{8}$ or $\sqrt[3]{23}$?

15. $2\sqrt[6]{6\frac{1}{2}}$ or $3\sqrt[3]{5}$?

16. $\sqrt[3]{6}$ or $\sqrt[4]{11}$?

Find the square root of

17. $33 + 20\sqrt{2}.$

18. $107 + 12\sqrt{77}.$

Simplify:

19. $\dfrac{x-1}{x+1}\left\{\dfrac{x-1}{\sqrt{x}-1}+\dfrac{1-x}{x+\sqrt{x}}\right\}.$

20. $\dfrac{\sqrt{1-x}+\dfrac{1}{\sqrt{1+x}}}{1+\dfrac{1}{\sqrt{1-x^2}}}.$ **21.** $\left\{\sqrt{\dfrac{x}{3}-2}-\sqrt{\dfrac{x}{3}+2}\right\}^2.$

22. $x\sqrt{\dfrac{x-y}{x+y}}+y\sqrt{\dfrac{x+y}{x-y}}-(3y^2-x^2)\sqrt{\dfrac{1}{x^2-y^2}}.$

Solve:

23. $2+\sqrt{x+3}=\sqrt{x-2}+3.$

24. $\dfrac{4}{\sqrt{x-4}}-\sqrt{x}-\sqrt{x-4}=0.$

25. $\dfrac{\sqrt{x}+4}{2\sqrt{x}-1}=\dfrac{\sqrt{x}+2}{2\sqrt{x}-3}.$

26. $\sqrt{\dfrac{2x}{3}+1}+\sqrt{\dfrac{2x}{3}-1}=\sqrt{\dfrac{8x}{3}-2}.$

Find the value of each of the following in the shortest way:

27. $(\sqrt{a+x})(\sqrt{a+x})(\sqrt{a-x})(\sqrt{a-x}).$

28. $(2\sqrt{x}-\sqrt{y})(2\sqrt{x}-\sqrt{y})(2\sqrt{x}+\sqrt{y})(2\sqrt{x}+\sqrt{y}).$

29. Simplify $\dfrac{(x+\sqrt{4-x^2})(x-\sqrt{4-x^2})}{(x+\sqrt{4-x^2})^2+(x-\sqrt{4-x^2})^2}.$

30. Show that $\dfrac{\sqrt{a^2-x^2}+\dfrac{x^2}{\sqrt{a^2-x^2}}}{a^2-x^2}$ reduces to $\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}.$

31. Show that $\dfrac{\sqrt{2ax-x^2}-\dfrac{(x-a)(a-x)}{\sqrt{2ax-x^2}}}{2ax-x^2}$ reduces to $\dfrac{a^2}{(2ax-x^2)^{\frac{3}{2}}}.$

32. Show that $\dfrac{\dfrac{-x-(a-x)}{x^2}}{2\sqrt{\dfrac{a-x}{x}}}$ reduces to $-\dfrac{a}{2x\sqrt{ax-x^2}}.$

33. Find the value of each of the following: $6 - \frac{2+\sqrt{3}}{5}$, $\frac{1}{3} \div \frac{2}{3}\sqrt{2}$, $\frac{1}{\frac{1}{3}\sqrt{8}}$, $\frac{\frac{1}{3}}{\frac{2}{3}\sqrt{2}}$, $\frac{1}{\sqrt{\frac{1}{5}}}$, $\frac{5}{2} - \frac{5}{3}\left(\frac{7-\sqrt{3}}{5}\right)$, $\frac{1}{1-\frac{1}{2}\sqrt{2}}$.

34. Find the value of $\frac{a}{1-r}$ when $a = 2$ and $r = \sqrt{2} + 1$. Also when $a = \frac{2}{3}$ and $r = \sqrt{2} - 2$.

35. Simplify $2\sqrt[6]{9} + \sqrt{50} - 6\sqrt{\frac{1}{2}} + \sqrt[3]{24} - 2\sqrt[9]{27} - 2\sqrt[4]{64}$.

36. Simplify $\dfrac{27\sqrt{3} - \dfrac{1}{\sqrt{3}}}{\sqrt{3}-1}$. Also $\dfrac{\dfrac{(\sqrt{3})^8}{\sqrt{3}} - \dfrac{1}{\sqrt{3}}}{\sqrt{3}-1}$.

37. Find the value of the following to three decimal places by extracting only one square root:

$$\sqrt{28} + 3\sqrt{\tfrac{1}{7}} - \sqrt{\tfrac{4}{7}} + 2\sqrt{63}.$$

38. Make up and work an example similar to Ex. 37.

39. Does $\sqrt{9 \times 16}$ equal 3×4? Does $\sqrt{9+16}$ equal $3 + 4$?

40. Does $\sqrt{a^2 + b^2}$ equal $\sqrt{a^2} + \sqrt{b^2}$? Can you prove your statement by an algebraic method? Does $\sqrt{a^2b^2}$ equal $\sqrt{a^2}\sqrt{b^2}$?

41. By use of radicals factor $x^2 - 2$. $x^2 - 5$. $x - 2$. $x - 5$. $x^2 + 2x - 2$.

42. Simplify $\frac{x^3 - 2\sqrt{2}}{x^2 - 2}$. Also $\frac{x^2 - 5}{x - \sqrt{5}}$.

43. Solve $\frac{1}{x} + \frac{1}{2x-y} = 5$, $\frac{4}{x} - \frac{3}{2x-y} = 6$.

44. In transposing a term in an equation, why do we change the sign of the term?

45. Solve

$$3\sqrt{x} + 5\sqrt{y} = 11.$$
$$6\sqrt{x} - 3\sqrt{y} = 9.$$

46. Simplify $\frac{a^{\frac{3}{2}} + ab}{ab - b^3} - \frac{\sqrt{a}}{\sqrt{a} - b}$.

47. Having given $\sqrt{3} = 1.732+$, find the value of $\sqrt{.03}$ to three decimal places in the shortest way. Also $\sqrt{27}$. $\sqrt{.27}$.

48. Factor $x^2 + a^2$ by the use of radicals (see Art. 101).

49. In like manner factor x^2+1. x^4+a^4.

50. How is the product of two monomial radicals obtained, if the radicals are of different orders? Give an example.

51. Show which is greater, $2\sqrt{3}$ or $3\sqrt[3]{2}$.

52. Show that $3\sqrt{3\sqrt{2}}$ equals $3\sqrt[4]{18}$.

53. Simplify $3\sqrt[6]{x^3y^3}+2\sqrt[4]{x^2y^2}-4\sqrt{4xy}+xy\sqrt[10]{\dfrac{1}{x^5y^5}}$.

54. Also $4\sqrt{18}+2\sqrt{\frac{1}{8}}-\sqrt{.5}+\sqrt{12\frac{1}{2}}$.

55. If $x=\dfrac{1+\sqrt{3}}{2}$, find the value of x^2+3x+2.

56. Find the value of $\sqrt{ab}\cdot\sqrt[3]{ab^2}\cdot\sqrt[4]{a^2b}$.

57. Multiply $\sqrt[3]{2}+2\sqrt[4]{2}$ by $\sqrt{\frac{1}{2}}$.

58. Collect as many instances as you can of men who have excelled in the study of mathematics and who have become distinguished as statesmen.

59. Distinguished as generals in war. As scientists. As writers.

EXERCISE 108

Oral

Reduce to simplest form:

1. $\sqrt{\dfrac{1}{3}}, \dfrac{1}{\sqrt{3}}, \sqrt[9]{8}, \sqrt{\dfrac{2}{3}}, \dfrac{2}{\sqrt{3}}, \dfrac{1}{\sqrt{a}}, \sqrt{\dfrac{1}{a}}\ \sqrt{\dfrac{a}{b}}, \sqrt[4]{\dfrac{9}{16}}$.

2. $\dfrac{1}{\sqrt{2}-1}, \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}, \dfrac{1}{2\sqrt{2}+3}$.

3. $\frac{1}{3}+\sqrt[4]{9}, 2\sqrt{3}+\sqrt[6]{27}, \sqrt[6]{4}+\sqrt[4]{25}, 2-\sqrt[4]{4}$.

4. $\sqrt{18}\div\sqrt{6}$, $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$, $\sqrt{4+\sqrt{5}}\times\sqrt{4-\sqrt{5}}$.

5. $(\sqrt{2})^6, (\sqrt{2})^5, (\sqrt{2})^7, (\sqrt{2})^1, (\sqrt{3})^1$.

6. Which is greater, $\sqrt{2}$ or $\sqrt[3]{3}$? $\sqrt[3]{4}$ or $\sqrt{3}$?

7. Solve $\sqrt{x+1}=3$.

8. Give the product of $\sqrt{x+y}+\sqrt{x-y}$ and $\sqrt{x+y}-\sqrt{x-y}$. Of $3\sqrt{8}$ and $5\sqrt{2}$. Of $3\sqrt{2}$ and $4\sqrt[3]{3}$.

9. Expand $(\sqrt{3}-\sqrt{2})^2$. Also $(\sqrt{x}+\sqrt{8+x})^2$.

10. Simplify: $2\sqrt{\frac{1}{2}}$, $3\sqrt{\frac{1}{2}}$, $3\sqrt{\frac{4}{3}}$, $5\sqrt{\frac{1}{5}}$, $\sqrt{\frac{2}{5}}$, $2\sqrt{\frac{9}{5}}$.

11. Factor $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}$, $1+\frac{2}{x}+\frac{1}{x^2}$, $\frac{x^2}{y^2}+2+\frac{y_2}{x^2}$, $a^3-\frac{1}{27}$.

12. What is meant by the root of an equation? The solution of an equation?

13. In adding fractions we retain the common denominator, but in clearing an equation of fractions we drop the common denominator. Why this difference?

14. In what respect is $\frac{\sqrt{15}}{3}$ a simpler expression than $\frac{\sqrt{5}}{\sqrt{3}}$?

At sight give the value of x in

15. $3+\sqrt{x}=4$.

16. $\sqrt{x+1}=4$.

17. $\sqrt{x+a}=b$.

18. $a+\sqrt{x}=c$.

19. $\sqrt{\frac{x}{a}}=b$.

20. $\sqrt{\frac{a}{x}}=b$.

21. $\sqrt{5x}=a$.

22. $\sqrt{ax}=c$.

23. $\sqrt{\frac{a}{x}}=\frac{b}{c}$.

CHAPTER XVIII

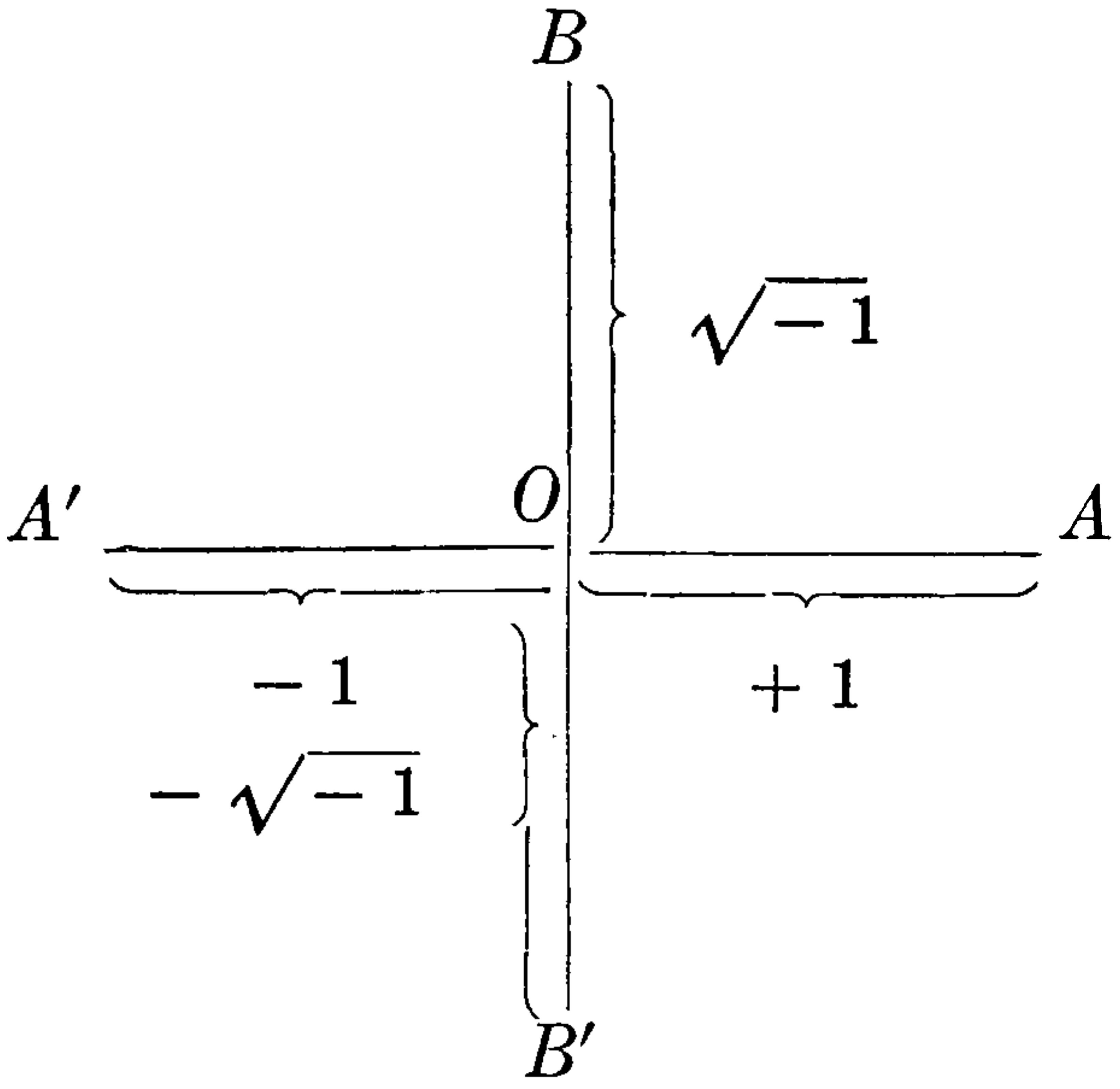
IMAGINARY QUANTITIES

218. An Imaginary Quantity is an indicated even root of a negative quantity; as $\sqrt{-4}$, $\sqrt{-3}$, and $\sqrt{-a}$.

The term "imaginary" is used because, so long as we confine ourselves to plus quantity and to its direct opposite, minus quantity, there is no number which multiplied by itself will give a negative number, as -4, for instance. All the quantity considered hitherto, that is, all positive or negative quantity, whether it is rational or irrational, is called **real quantity**.

If we extend the realm of quantity outside of positive and negative quantity, imaginary numbers are as real as any others, as will be shown in the next article.

A **complex number** is a number part real and part imaginary; as $3 + 2\sqrt{-1}$ and $a + b\sqrt{-1}$.



219. Meaning of $\sqrt{-1}$.

If $OA = +1$, and OA' is of the same length, but lying in the opposite direction from O, $OA' = -1$.

Hence, we regard the operation of converting a plus quantity into negative quantity as equivalent to a rotation through an angle of 180°. If we divide this rotation into two equal rotations, each of these will be a rotation through 90°.

Hence, $\sqrt{-1}$ must be equivalent (geometrically) to the result of rotating the plus unit of quantity through 90°. Hence, $\sqrt{-1}$ on our figure will be represented by OB.

Hence, it is easy to see, also, that $\sqrt{-1} \times \sqrt{-1} = -1$.

We thus perceive that the introduction of imaginary quantity enlarges the field of quantity considered in algebra from mere quantity in a line to quantity in a plane. This gives a vast extension to power of algebraic processes and introduces many economies in them, as will be found by the student who pursues the study of mathematics extensively.

In taking up the subject for the first time, we consider only a few of the first properties of imaginaries, so called.

220. The Fundamental Principle in treating imaginaries is that $\sqrt{-1} \times \sqrt{-1} = -1$.

Using i as a symbol for $\sqrt{-1}$, this principle is, $i \times i = -1$, or $i^2 = -1$.

Considering this matter algebraically, if we use the law of signs in the most general form,

$$(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = \sqrt{1} = \pm 1$$

Now, if we extract the square root of $+1$, we shall not have $\sqrt{-1}$. But if we extract the square root of -1, we shall have $\sqrt{-1}$.

Hence, we must limit the product $\sqrt{-1} \times \sqrt{-1}$ to -1.

Likewise, $\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1}$

$$= \sqrt{a}\sqrt{b}(\sqrt{-1})^2 = -\sqrt{ab}$$

Or, using the symbol i, $ai \times bi = -ab$.

221. Powers of $\sqrt{-1}$.

$(\sqrt{-1})^1 = \sqrt{-1}$ $\quad \therefore i = \sqrt{-1}$

$(\sqrt{-1})^2 = -1$ $\quad \therefore i^2 = -1$

$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1}$ $\quad \therefore i^3 = -i$ (See OA' of figure in Art. 219.)

$(\sqrt{-1})^4 = (\sqrt{-1})^3\sqrt{-1} = +1$ $\quad \therefore i^4 = 1$

Thus, the first four powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$; and for the higher powers, as the fifth, sixth, etc., these four results recur regularly. The same fact is clear from the figure in Art. 219.

222. Operations with Imaginaries. It follows from Art. 220 that, in performing operations with imaginaries, we *use all the ordinary laws of algebra,* with the exception of a limitation in the use of signs, which may be mechanically stated as follows:

The product of two minus signs under the radical sign of the second degree gives a minus sign outside the radical sign. But in dividing *first indicate the division and afterwards rationalize the denominator.*

Ex. 1. Add $\sqrt{-9}$, $-3+2\sqrt{-1}$, $7-2\sqrt{-16}$.

$$\begin{aligned} \sqrt{-9} &= 3\sqrt{-1} \\ -3+2\sqrt{-1} &= -3+2\sqrt{-1} \\ 7-2\sqrt{-16} &= \underline{7-8\sqrt{-1}} \\ &\ 4-3\sqrt{-1} \quad \textit{Sum} \end{aligned}$$

Ex. 2. Multiply $2\sqrt{-3}+3\sqrt{-6}$ by $3\sqrt{-3}-5\sqrt{-2}$.

$$\begin{array}{l} 2\sqrt{-3}+3\sqrt{-6} \\ \underline{3\sqrt{-3}-5\sqrt{-2}} \\ 6(-3)-9\sqrt{18} \\ \underline{\qquad\qquad +10\sqrt{6}+15\sqrt{12}} \\ -18-27\sqrt{2}+10\sqrt{6}+30\sqrt{3} \quad \textit{Product} \end{array}$$

Ex. 3. Divide $-2\sqrt{6}$ by $\sqrt{-2}$.

$$\frac{-2\sqrt{6}}{\sqrt{-2}} = \frac{-2\sqrt{6}}{\sqrt{-2}} \times \frac{\sqrt{-2}}{\sqrt{-2}}$$

$$= \frac{-2\sqrt{-12}}{-2} = 2\sqrt{-3} \quad \textit{Quotient}$$

Ex. 4. Extract the square root of $1+4\sqrt{-3}$.

$$1+4\sqrt{-3} = 1+2\sqrt{-12}$$

The two numbers which multiplied together give -12, and added together give 1, are 4 and -3.

$$\therefore \sqrt{1+4\sqrt{-3}} = \sqrt{4} + \sqrt{-3} = 2 + \sqrt{-3}\ \textit{Root}$$

Let the pupil work the above examples using i instead of -1.

EXERCISE 109

Collect:

1. $7\sqrt{-4} + 3\sqrt{-49} - 10\sqrt{-9}$.
2. $\sqrt{-12} - \sqrt{-27} + 2\sqrt{-3} + \sqrt{-75}$.
3. $5\sqrt{-\frac{1}{2}} - 3\sqrt{-\frac{9}{2}} + 4\sqrt{-50} - \sqrt{-200}$.
4. $2\sqrt{-a^2} - 3a\sqrt{-4} + \dfrac{2}{a}\sqrt{-16a^4} - \frac{2}{3}\sqrt{-36a^2}$.
5. $a + b\sqrt{-1} - b - a\sqrt{-1} - \sqrt{-b^2} + 2\sqrt{-a^2} - a$.
6. $(a-2b)\sqrt{-1} - (2a+b)\sqrt{-1}$.
7. Express in terms of i the results obtained in Exs. 1–6.

Multiply:

8. $\sqrt{-1}$ by $\sqrt{-2}$.
9. $\sqrt{-5}$ by $-2\sqrt{-5}$.
10. $-\sqrt{-7}$ by $-\sqrt{\;}$.
11. $-\sqrt{-12}$ by $-\sqrt{{}^{7}-18}$.
12. $2\sqrt{-14}$ by $-2\sqrt{-21}$.
13. $-5\sqrt{-2}$ by $-2\sqrt{-5}$.
14. $-\sqrt{x-y}$ by $\sqrt{y-x}$.
15. $-a\sqrt{1-a}$ by $-\sqrt{(a-1)^3}$.
16. $\sqrt{-1} + \sqrt{-2}$ by $\sqrt{-1} - 2\sqrt{-2}$.
17. $3\sqrt{-3} - 2\sqrt{-2}$ by $2\sqrt{-3} + 3\sqrt{-2}$.
18. $2\sqrt{2} - 2\sqrt{-2}$ by $3\sqrt{2} + 3\sqrt{-2}$.
19. $3\sqrt{-3} - \sqrt{-2}$ by $2\sqrt{3} - \sqrt{-6}$.
20. $\sqrt{-1} - \sqrt{-2} + \sqrt{-3}$ by $\sqrt{-1} + \sqrt{-2} + \sqrt{-3}$.
21. $\sqrt{-3} + \sqrt{-2}$ by $\sqrt{3} + \sqrt{-2}$.
22. $x - 2 + \sqrt{-3}$ by $x - 2 - \sqrt{-3}$.
23. $a\sqrt{-a} + b\sqrt{-b}$ by $a\sqrt{-a} - b\sqrt{-b}$.
24. $x - 1 - \sqrt{-1}$ by $x - 1 + \sqrt{-1}$.

25. Multiply $x - \frac{1-\sqrt{-3}}{2}$ by $x - \frac{1+\sqrt{-3}}{2}$.

26. In the shortest way find the value of

$$(\sqrt{-3}+\sqrt{-2})\,(\sqrt{3}+\sqrt{2})\,(\sqrt{-3}-\sqrt{-2})\,(\sqrt{3}-\sqrt{2}).$$

Divide:

27. $-\sqrt{18}$ by $\sqrt{-6}$.

28. $-\sqrt{-12}$ by $-\sqrt{-2}$.

29. $-6\sqrt{-15}$ by $2\sqrt{-3}$.

30. $8\sqrt{-a^3}$ by $-2\sqrt{a}$.

31. $2\sqrt{-18}-4\sqrt{-15}+10\sqrt{30}$ by $-2\sqrt{-3}$.

32. $a\sqrt{-a}-2a\sqrt{-6a}-a^2\sqrt{3a^3}$ by $-a\sqrt{-a}$.

Express with rational denominators:

33. $\dfrac{1}{3-\sqrt{-2}}$.

34. $\dfrac{2-\sqrt{-3}}{2+\sqrt{-3}}$.

35. $\dfrac{3\sqrt{2}-\sqrt{-2}}{2\sqrt{2}+\sqrt{-2}}$.

36. $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}$.

37. $\dfrac{3\sqrt{-5}-\sqrt{-2}}{2\sqrt{-5}-3\sqrt{-2}}$.

38. $\dfrac{\sqrt{x-1}+\sqrt{1-x}}{2\sqrt{1-x}-\sqrt{x-1}}$.

39. $\dfrac{(1-\sqrt{-1})^2}{2-3\sqrt{-1}}$.

40. $\dfrac{3\sqrt{2}+2\sqrt{-7}-\sqrt{-10}}{3\sqrt{2}-2\sqrt{-7}+\sqrt{-10}}$.

Find the square root of

41. $3-6\sqrt{-6}$.

42. $1-2\sqrt{-6}$.

43. $12\sqrt{-6}-6$.

44. $12\sqrt{10}-38$.

45. $-29-24\sqrt{-5}$.

46. $7+40\sqrt{-2}$.

47. By use of i (or $\sqrt{-1}$), factor a^2+b^2. Also a^4+b^4. x^2+2. $a^2x^2+b^2$. x^2+1. x^8+1.

48. For what values of x is $\sqrt{2-x}$ imaginary? $\sqrt{2-x^2}$?

49. Find the value of $(\sqrt{-1})^5$. Of $(-\sqrt{-1})^4$, $(\sqrt{-1})^6$, $(-\sqrt{-1})^7$, $(\sqrt{-1})^{-3}$, $(-\sqrt{-1})^{-9}$, $(\sqrt{-1})^{\frac{2}{3}}$.

Ex. Simplify $3(\sqrt{-1}+2)^2-(2\sqrt{-1}-1)^2$.

Substitute i for $\sqrt{-1}$.

$$\begin{aligned} 3(i+2)^2-(2i-1)^2 &= 3i^2+12i+12-4i^2+4i-1 \\ &= -i^2+16i+11 \\ &= -(-1)+16\sqrt{-1}+11 \\ &= 12+16\sqrt{-1} \; \textit{Ans.} \end{aligned}$$

Simplify:

50. $i^2+3i^4-2i^8$.

51. $i^3-5i+4i^4$.

52. $i^3\times i^4\times 3i^5$.

53. $(i-1)^3-(i-1)^2+3(i-1)$.

54. $(\sqrt{-1}-1)^3-(\sqrt{-1}-1)^2+2(\sqrt{-1}-1)$.

55. $(\sqrt{-1}-2)(3\sqrt{-1}+1)-(\sqrt{-1}-3)^2-(\sqrt{-2})^2$.

56. $(\sqrt{-1}-1)^4+3(\sqrt{-1}-1)^3+4(\sqrt{-1}-1)^2$.

57. If $x=\frac{1-\sqrt{-1}}{2}$, find the value of $3x^2-6x+7$. Of x^3-5x^2+2x-1.

58. Simplify i^{4q+1}. Also i^{4q+2}, i^{4q+3}, i^{4q+4}.

59. Who first discussed imaginary quantities and when? Who first put the use of these quantities on a scientific basis? Who invented the symbol i for $\sqrt{-1}$?

60. Make up and work an example similar to Ex. 2. To Ex. 17. Ex. 51.

EXERCISE 110

ORAL

1. State which of the following are imaginary: $\sqrt{-3}$, $-\sqrt{\frac{1}{2}}$, $-\sqrt{\frac{2}{3}}$, $\sqrt{-\frac{2}{3}}$, $\sqrt[3]{-8}$, $\sqrt[4]{-16}$, $\sqrt[5]{-1}$.

2. Give the product of each of the following: $\sqrt{-2}\times\sqrt{-3}$, $\sqrt{-2}\times\sqrt{3}$, $-\sqrt{-2}\times\sqrt{3}$, $\sqrt{-a}\times\sqrt{-a}$, $-\sqrt{a}\times\sqrt{-a}$, $-\sqrt{3}\times\sqrt{-5}$, $i^3\times i^4$, $-i^2\times i^3$.

3. Give the product of $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}-\sqrt{-2})$, $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$, $(5+\sqrt{-2})(5-\sqrt{-2})$, $(\frac{1}{2}+\frac{1}{2}\sqrt{-2})(\frac{1}{2}-\frac{1}{2}\sqrt{-2})$.

4. Of $3i\times 2i$, $-5i\times 3i$, $-2i^2\times 3i$, $-4i^2\times 2i^2$.

5. Does $(a+x)^n$ equal a^n+x^n?

6. Does $\sqrt[n]{a+x}$ equal $\sqrt[n]{a}+\sqrt[n]{x}$?

7. Is $(ax)^n$ equal to a^nx^n?

8. Is $\sqrt{ax}$ equal to $\sqrt[n]{a}\sqrt[n]{x}$?

9. What advantage is it to know principle contained in Ex. 7?

10. That contained in Ex. 8?

11. Which is greater, $(3^5)^{10}$ or $3^5\cdot 3^{10}$? How many times greater?

At this point, if the teacher prefers, the class may review by use of Exercise 157 (p. 475).

CHAPTER XIX

QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY

223. Need and Utility of Equations of the Second Degree.

Ex. One basketball team has won 5 games out of 17 played, and another team has won 6 games out of 12 played. How many straight (i. e. consecutive) games must the first team win from the second in order that their averages of games won may be equal?

Let x = the required number of games.

Then $\dfrac{5+x}{17+x} = \dfrac{6}{12+x}$

Simplifying this equation we obtain,

$$x^2 + 11x = 42$$

Hence, in order to find the value of x, it will be necessary to solve an equation of the second degree.

Why do we now proceed to make definitions and rules?

224. **A quadratic equation** of one unknown quantity is an equation containing the second power of the unknown quantity, but no higher power.

A **pure quadratic equation** is one in which the second power of the unknown quantity occurs, but not the first power.

Ex. $5x^2 - 12 = 0$.

A pure quadratic equation is sometimes termed an incomplete quadratic equation.

An **affected** (or complete) **quadratic equation** is one in which both the first and second powers of the unknown quantity occur.

Ex. $3x^2 - 7x + 12 = 0$.

PURE QUADRATIC EQUATIONS

225. Solution of Pure Quadratics. Since only the second power, x^2, of the unknown quantity occurs in a pure quadratic equation, in solving such an equation, we

Reduce the given equation to the form $x^2 = c$;

Extract the square root of both members.

Ex. 1. Solve $\dfrac{x^2 - 12}{3} = \dfrac{x^2 - 4}{4}$.

Clearing of fractions, $4x^2 - 48 = 3x^2 - 12$

Hence, $x^2 = 36$

Extracting the square root of each member,

$$x = +6, \quad \text{or} \quad -6.$$

That is, since the square of $+6$ is 36, and also the square of -6 is 36, x has two values, either of which satisfies the original equation. These two values of x are best written together. Thus,

$$x = \pm 6 \quad \textit{Roots}$$

CHECK. For $x = 6$

$$\frac{x^2 - 12}{3} = \frac{36 - 12}{3} = \frac{24}{3} = 8$$

$$\frac{x^2 - 4}{4} = \frac{36 - 4}{4} = \frac{32}{4} = 8$$

CHECK. For $x = -6$

$$\frac{x^2 - 12}{3} = \frac{36 - 12}{3} = 8$$

$$\frac{x^2 - 4}{4} = \frac{36 - 4}{4} = 8$$

Ex. 2. Solve $\dfrac{a}{x^2 - b} = \dfrac{b}{x^2 - a}$.

$$ax^2 - a^2 = bx^2 - b^2$$

$$ax^2 - bx^2 = a^2 - b^2$$

$$x^2 = a + b$$

$$x = \pm \sqrt{a + b} \quad \textit{Roots}$$

Let the pupil check the work.

EXERCISE 111

Solve:

1. $5x^2 = 80.$

2. $3x^2 - 5 = x^2 + 3.$

3. $\frac{7}{3}x^2 - 1 = \frac{1}{3} - 3x^2.$

4. $1 - \frac{2}{5}x^2 = x^2 - 4\frac{3}{5}.$

5. $\dfrac{5x^2}{3} = 1 - \dfrac{x^2}{3}.$

6. $\dfrac{x^2 - 5}{7} = \dfrac{\frac{2}{7} - x^2}{5}.$

7. $\dfrac{3 - x^2}{11} + \dfrac{x^2 + 5}{6} = 3.$

8. $\dfrac{3}{4x^2} - \dfrac{1}{3x^2} = \dfrac{5}{12}.$

9. $\dfrac{1}{2x - 1} - \dfrac{1}{2x + 1} = 3\frac{5}{9}.$

10. $ax^2 + a^3 = 5a^3 - 3ax^2.$

11. $ax^2 + c = b.$

12. $\dfrac{x + 2a}{x - 2a} + \dfrac{x - 2a}{x + 2a} = \dfrac{26}{5}.$

13. $\dfrac{2c}{x - c} + \dfrac{5x + 2c}{3x} + 1 = 0.$

14. $(ax + b)^2 + (ax - b)^2 = 10b^2.$

15. $(x + a)(x - b) + (x - a)(x + b) = 2(a^2 + b^2 + ab).$

16. $3(2x - 5)(x + 1) - 2(3x + 2)(2x - 3) = x - 9.$

17. If $x^2 = \dfrac{28}{1 + 3v}$, find the value of x when $v = -\dfrac{2}{7}.$

18. The square of a certain number increased by 9 equals twice the square of the same number diminished by 27. Find the number.

19. State Ex. 1 as a problem concerning a number.

20. Also Ex. 2.

21. Also Ex. 3.

22. The product of a number by one third of itself equals 12. Find the number.

23. A certain field contains 256 sq. rd. and the field is four times as long as it is wide. Find the dimensions of the field.

24. A certain field is four times as long as it is wide. If each of its sides is increased by one half, its area is increased 180 sq. rd. Find the dimensions of the field.

25. Who first formed the idea of absolute or independent negative numbers (see p. 461). How was negative number used before this? How did the Arabs treat it?

26. Make up and work an example similar to Ex. 2. To Ex. 14. Ex. 18. Ex. 23.

27. Practice oral work with fractions as in Exercise 58 (p. 190).

AFFECTED QUADRATIC EQUATIONS

226. Completing the Square. An affected quadratic equation may in every instance be reduced to the form

$$x^2 + px = q$$

An equation in this form may then be solved by a process called *completing the square.* This process consists in adding to both members of the equation such a number as will make the left-hand member a perfect square. The use of familiar elementary processes then gives the values of x.

Thus, to solve $\qquad x^2 + 6x = 16$

take half the coefficient of x (that is, 3), square it, and add the result (that is, 9) to both members of the original equation. We obtain

$$x^2 + 6x + 9 = 25$$

or, $\qquad (x + 3)^2 = 5^2$

Extract the square root of both members,

$$x + 3 = \pm 5$$

Hence, $\qquad x = -3 \pm 5$

$$\begin{array}{ll} \text{That is,} & x = -3 + 5 = 2 \\ \text{Also,} & x = -3 - 5 = -8 \end{array}\Bigg\} \text{ Roots}$$

Hence we have the general rule:

By clearing the given equation of fractions and parentheses, transposing terms, and dividing by the coefficient of x^2, *reduce the given equation to the form* $x^2 + px = q$;

Add the square of half the coefficient of x to each member of the equation;

Extract the square root of each member;

Solve the resulting simple equations.

Before clearing an equation of fractions, it is important to *reduce each fraction to its simplest form.*

Ex. 1. Solve $6x^2 - 14x = 12$.

Dividing by 6, $x^2 - \frac{7}{3}x = 2$

Completing the square, $x^2 - \frac{7}{3}x + (\frac{7}{6})^2 = 2 + \frac{49}{36} = \frac{121}{36}$

Extracting the square root, $x - \frac{7}{6} = \pm \frac{11}{6}$

$$x = \frac{7}{6} \pm \frac{11}{6}$$

$x = 3$, or $-\frac{2}{3}$ *Roots*

CHECK. For $x = 3$	CHECK. For $x = -\frac{2}{3}$
$6x^2 - 14x = 6 \times 9 - 14 \times 3$	$6x^2 - 14x = 6 \times \frac{4}{9} + 14 \times \frac{2}{3}$
$= 54 - 42 = 12$	$= \frac{8}{3} + \frac{28}{3} = 12$

Ex. 2. Solve $3x^2 = 2(1 + 2x)$.

Clearing, $3x^2 = 2 + 4x$

Transposing, $3x^2 - 4x = 2$

Hence, $x^2 - \frac{4}{3}x = \frac{2}{3}$

$$x^2 - \frac{4}{3}x + (\frac{2}{3})^2 = \frac{10}{9}$$

$$x = \frac{2}{3} \pm \sqrt{\frac{10}{9}} = \frac{2 \pm \sqrt{10}}{3}$$ *Roots*

Let the pupil supply the check.

Ex. 3. Solve $\frac{2x - 2}{x^2 - 1} = 3$.

If we fail to reduce the fraction $\frac{2x - 2}{x^2 - 1}$ to its simplest form before clearing the equation of fractions, we obtain

$$2x - 2 = 3x^2 - 3$$

$$3x^2 - 2x - 1 = 0$$

Whence, $x = -\frac{1}{3}, 1$

CHECK. For $x = -\frac{1}{3}$

$$\frac{2x-2}{x^2-1} = \frac{-\frac{2}{3}-2}{\frac{1}{9}-1} = 3$$

CHECK. For $x = 1$

$$\frac{2x-2}{x^2-1} = \frac{2-2}{1-1} = \frac{0}{0}$$

Hence, $x = 1$ is not a root.

The root $x = 1$ was introduced (see Art. 129) by a failure to reduce the given fraction to its simplest form.

Let the pupil give the correct solution of Ex. 3.

EXERCISE 112

Solve:

1. $x^2 + 10x = 24.$

2. $x^2 - 8x - 20 = 0.$

3. $x^2 - 5x = 6.$

4. $x^2 + 11x + 24 = 0.$

5. $3x^2 + 4x = 7.$

6. $5x^2 - 6x = 8.$

7. $2x^2 - 5x = 7.$

8. $3x^2 + 7x = 26.$

9. $2x + 3\frac{3}{4}x^2 = 4.$

10. $35 = 2x^2 - 3x.$

11. $3x^2 = \frac{1}{3}x + 2\frac{2}{3}.$

12. $\frac{1}{2} = \frac{7x}{6} + x^2.$

13. $\frac{x-1}{2} + \frac{2}{x-1} = 2\frac{1}{2}.$

14. $\frac{3x+5}{x+4} = 3 - \frac{2x-5}{x-2}.$

15. $\frac{2x-1}{x+3} - \frac{x+1}{2x-3} = -\frac{1}{2}.$

16. $x - \frac{1}{x} = 4 - \frac{x}{2} + \frac{1}{2x}.$

17. $\frac{x}{5-x} - \frac{5-x}{x} = \frac{15}{4}.$

18. $\frac{1}{x+2} - \frac{x}{x-2} = \frac{4}{3}.$

19. $\frac{2x+1}{x-1} + x = \frac{x-1}{3}.$

20. $\frac{2}{3x-1} + \frac{3x}{2x-5} = 0.$

21. $\frac{3x+1}{5x+4} = \frac{x+3}{4x+5}.$

22. $\frac{1}{2}(x+1) - \frac{x}{3}(2x-1) = -12.$

23. $\frac{x-3}{2x} + \frac{3x-1}{3} = 1 - \frac{1-x}{4x} + \frac{3+2x}{6}.$

24. $\frac{2x-1}{x+1} - \frac{3x-4}{x-1} = 1 - \frac{4x-14}{1-x^2}.$

25. $\dfrac{x-3}{3x-2}+\dfrac{5x-7}{4-9x^2}=\dfrac{2x+5}{2+3x}-1.$

26. $\dfrac{2x-1}{x+1}-\dfrac{1-3x}{x+2}-\dfrac{x-7}{x-1}=4.$

27. $x^2+2x=1.$

29. $2x^2+5x=-4.$

28. $3x^2-5x=-1.$

30. $9x^2-6x+5=0.$

31. $3x(x+1)-(x-2)(x+3)=2+(1-x)^2.$

32. $\dfrac{x^2+x-1}{x^2+x+1}+\dfrac{x^2-x-1}{x^2-x+1}=2.$

33. $\dfrac{4x-2}{1-x}=3+\dfrac{x^2+1}{1-x}.$

34. $x+\dfrac{2x-8}{x-3}=9-\dfrac{3x-7}{x-3}.$

35. $11-\dfrac{8x+4}{x+2}=\dfrac{7x+20}{x+2}-x+1.$

36. $\dfrac{7x-69}{4(x+5)(x-8)}+\dfrac{2}{(x+4)(x+5)}=\dfrac{1}{x+4}.$

37. Take the equation $\dfrac{3x-3}{x^2-1}-5=0$ and, by separating the fraction into two parts, form an equation which contains an extraneous root, the appearance of the equation giving no indication of this fact.

38. Make up and work an example similar to Ex. 37.

39. The square of a certain number increased by 4 times the number equals 45. Find the number.

40. State Ex. 1 as a problem concerning numbers. Ex. 7.

41. Find two consecutive numbers such that the sum of their squares is 41.

42. Practice the oral solution of equations as in Exercise 64 (p. 209).

227. Literal Quadratic Equations are solved by the methods employed in solving quadratic equations with numerical coefficients.

Ex. 1. Solve $\frac{x^2}{2} - \frac{ax}{6} = \frac{a}{6}(x + a)$.

Clearing, $3x^2 - ax = ax + a^2$

Hence, $3x^2 - 2ax = a^2$

$$x^2 - \frac{2a}{3}x = \frac{a^2}{3}$$

$$x^2 - (\ ) + \left(\frac{a}{3}\right)^2 = \frac{4a^2}{9}$$

$$x = a,\ -\frac{a}{3} \quad \textit{Roots}$$

Let the pupil check the work.

Ex. 2. Solve $(a - b)^2x^2 - (a^2 - b^2)x = -ab$.

$$x^2 - \frac{a + b}{a - b}x = -\frac{ab}{(a - b)^2}$$

$$x^2 - (\ ) + \frac{1}{4}\left(\frac{a + b}{a - b}\right)^2 = \frac{1}{4}\left(\frac{a + b}{a - b}\right)^2 - \frac{4ab}{4(a - b)^2}$$

$$x - \frac{a + b}{2(a - b)} = \pm\frac{a - b}{2(a - b)}$$

$$x = \frac{a}{a - b},\quad \frac{b}{a - b} \quad \textit{Roots}$$

Let the pupil check the work.

EXERCISE 113

Solve:

1. $x^2 + 4ax = 12a^2$.
2. $x^2 + 4bx = 21b^2$.
3. $x^2 + 3cx - 10c^2 = 0$.
4. $x^2 = 6a^2b^2 - 5abx$.
5. $6x^2 = 12b^2 + bx$.
6. $3x^2 + 4cdx = 15c^2d^2$.
7. $2a^2x^2 + ax = 3$.
8. $7c^2x^2 - 10acx + 3a^2 = 0$.
9. $x^2 + \frac{5x}{a} = \frac{6}{a^2}$.
10. $x^2 + \frac{bx}{a} = \frac{3b^2}{4a^2}$.

11. $x^2 - (a+1)x = -a$.

12. $2a^2x^2 + abx = 15b^2$.

13. $x^2 + 2x - 3ax = 6a$.

14. $3a^2x^2 + a(3b-5)x = 5b$.

15. $abx^2 + (a^2+b^2)x + ab = 0$.

16. $ax^2 - a^2x + x = a$.

17. $\frac{x}{a-x} + \frac{a+b}{x} = \frac{a}{a-x}$.

18. $x - \frac{1}{x-a} = a$.

19. $4(x^2-1) = b(4x-b)$.

20. $(a+b)x^2 - (a-b)x - \frac{ab}{a+b} = 0$.

21. $abx^2 = \frac{1}{ab}\left[x(a+b) - \frac{1}{ab}\right]$.

22. $\frac{x^2+1}{x} = \frac{a+b}{c} + \frac{c}{a+b}$.

23. $a(x^2-b^2) + b(x^2-b^2+c) + cx = 0$.

24. $(a+c)x^2 - (2a+c)x + a = 0$.

25. $\frac{x+1}{x^2} = \frac{a+1}{a^2}$.

26. $ax^2 + bx + c = 0$.

27. $x^2 + px + q = 0$.

28. Show that the sum of the roots obtained in Ex. 27 equals $-p$. Also that the product of these roots equals q.

29. How many examples in Exercise 45 (p. 155) can you now work at sight?

228. The Factorial Method of solving equations was explained in Art. 104. Sometimes this method must be supplemented by the method of completing the square.

Ex. 1. Solve $x^3 + 1 = 0$.

Factoring, $(x+1)(x^2-x+1) = 0$

$x + 1 = 0$, gives $x = -1$ *Root*.

Also, $x^2 - x + 1 = 0$

Whence, $x^2 - x = -1$

$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$ *Roots*

Let the pupil check the work.

The factorial method of solution is especially helpful in solving certain literal quadratic equations.

Ex. 2. Solve $(a-b)x^2-(a-b)x+ab=0$ by the factorial method.

We obtain $[(a-b)x-a][(a-b)-b]=0$

Hence, $x=\dfrac{a}{a-b}, \dfrac{b}{a-b}$ *Roots*

Let the pupil check the work.

This example is the same as Ex. 2 solved on p. 348. On comparing the solutions, we observe that at least three fourths of the labor of solution is saved by use of the factorial method.

EXERCISE 114

Solve:

1. $x^2+8x+7=0.$
2. $x^2-5x=84.$
3. $6x^2+7x=90.$
4. $3x^2-10x+3=0.$
5. $24x^2=2x+15.$
6. $3a^2x^2+10ax=8.$
7. $x^4=16.$
8. $x^3=8.$
9. $x^4-5x^2+4=0.$
10. $x^5-x^4-x+1=0.$
11. $(2x-1)(6x^2-x-2)=0.$
12. $3(x^2-1)-2(x+1)=0.$
13. $5(x^2-4)=3(x-2).$
14. $7(x^4-16)-53x(x^2-4)=0.$
15. $3x(x^2-1)+2(x-1)=0.$
16. $x^3-27=13x-39.$
17. $2x^3+2x^2=x+1.$
18. $2x^3+6x^2=3x^2+8x-3.$
19. Find the six roots of $x^6-1=0.$
20. Find all the values of $\sqrt[3]{1}.$
21. Find all the values of $\sqrt[3]{8}.$
22. Obtain a complete solution of the equation $x^{\frac{3}{2}}=8.$
23. Solve $x^2+(a+b)x+ab=0$ by the method of Art. 226.

Solve the same equation by the factorial method.

About how much shorter is the latter method than the former?

Solve by the factorial method:

24. $x^2 + cx + dx + cd = 0.$

25. $abcx^2 - (a^2b^2 + c^2)x + abc = 0.$

26. $\dfrac{x}{a - x} + \dfrac{a + b}{x} = \dfrac{a}{a - x}.$

27. $x^2 - 2bx + b^2 + x - b = 0.$

28. $a(b - c)x^2 + b(c - a)x + c(a - b) = 0.$

29. $(4a^2 - 9b^2)(x^2 + 1) = 2x(4a^2 + 9b^2).$

30. $\dfrac{a - b + x}{a + b + x} + \dfrac{a + b}{x + b} = 2.$

31. $\dfrac{a + 4b}{x + 2b} - \dfrac{a - 4b}{x - 2b} = \dfrac{4b}{a}.$

32. $\dfrac{x^2}{a + b} - \left(1 + \dfrac{1}{ab}\right)x + \dfrac{1}{a} + \dfrac{1}{b} = 0.$

Form the equations whose roots are

33. 2, 3.

34. $-2, -3.$

35. $-2, 3.$

36. 0, 2.

37. 2, 3, 4.

38. $\sqrt{2}, -\sqrt{2}, 0.$

39. Solve $(a + b)^2x^2 - (a^2 - b^2)x - ab = 0$ by the method of completing the square (Art. 226). Now solve it by the factorial method. Compare the work in the two processes. Why do we not solve all quadratic equations by use of the factorial method?

40. Find all the roots in the solution of $x^3 = 16x$.

How might one of these roots be lost by careless use of Ax. 4 (p. 95)?

41. Work again Exs. 62–68 of Exercise 75 (p. 246).

42. How many of the examples in Exercise 114 can you work at sight?

EQUATIONS IN THE QUADRATIC FORM

229. Simple Unknown Quantity. An equation containing only two powers of the unknown quantity, the index of one power being twice the index of the other power, is an equation of the quadratic form. It may be solved by the methods already given for affected quadratic equations.

Ex. 1. Solve $x^4 - 5x^2 = -4$.

Adding $(\frac{5}{2})^2$ to both members will make the left-hand member a perfect square. Thus,

$$x^4 - 5x^2 + (\tfrac{5}{2})^2 = \tfrac{9}{4}$$

Hence,
$$x^2 - \tfrac{5}{2} = \pm \tfrac{3}{2}$$
$$x^2 = 4, \quad \text{or } 1$$
$$x = \pm 2, \ \pm 1 \ \textit{Roots}$$

Let the pupil check the work.

This equation might also have been solved by the factorial method.

Ex. 2. Solve $2\sqrt[3]{x^{-2}} - 3\sqrt[3]{x^{-1}} = 2$.

Using fractional exponents,

$$2x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} = 2$$

Whence,
$$x^{-\frac{2}{3}} - \tfrac{3}{2}x^{-\frac{1}{3}} = 1$$
$$x^{-\frac{2}{3}} - (\ ) + \tfrac{9}{16} = \tfrac{25}{16}$$
$$x^{-\frac{1}{3}} - \tfrac{3}{4} = \pm \tfrac{5}{4}$$
$$x^{-\frac{1}{3}} = 2, \quad -\tfrac{1}{2}$$

Whence,
$$x^{\frac{1}{3}} = \tfrac{1}{2}, \quad -2$$
$$x = \tfrac{1}{8}, \quad -8 \ \textit{Roots}$$

Let the pupil check the work.

230. Compound Unknown Quantity. A polynomial may be used in place of a single quantity as an unknown quantity

Ex. 1. Solve $2\sqrt{x + 12} + 3\sqrt[4]{x + 12} = 6$.

This equation may be written, $2(x+12)^{\frac{1}{2}} + 3(x+12)^{\frac{1}{4}} = 6$

Let $(x+12)^{\frac{1}{4}} = y$; then $(x+12)^{\frac{1}{2}} = y^2$

Hence, substituting, $2y^2 + 3y = 6$

Whence, $y = 2$, or $-\frac{7}{2}$

$$\therefore \sqrt[4]{x+12} = 2 \qquad x + 12 = 16 \qquad x = 4 \text{ Root}$$

$$\sqrt{x+12} = -\frac{7}{2} \qquad x + 12 = \frac{2401}{16} \qquad x = 138\tfrac{1}{16} \text{ Root}$$

Let the pupil check the work.

Ex. 2. Solve $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$.

Add 18 to both sides,

$$x^2 - 7x + 18 + \sqrt{x^2 - 7x + 18} = 42$$

Let $\sqrt{x^2 - 7x + 18} = y$ then $y^2 + y = 42$

Whence, $y = 6$ or -7

Hence, $\sqrt{x^2-7x+18} = 6$

$$x^2-7x+18 = 36 \qquad x = 9, \quad -2 \text{ Roots}$$

$\sqrt{x^2-7x+18} = -7$

$$x^2-7x+18 = 49 \qquad x = \tfrac{1}{2}(7 \pm \sqrt{173}) \text{ Roots}$$

Let the pupil check the work.

EXERCISE 115

Solve:

1. $x^4 - 17x^2 + 16 = 0$.
2. $4x^4 - 13x^2 + 9 = 0$.
3. $27x^6 = 35x^3 - 8$.
4. $3x^{\frac{1}{2}} - 5x^{\frac{1}{4}} = 2$.
5. $27x^3 + 19x^{\frac{3}{2}} = 8$.
6. $3x^{\frac{2}{3}} = 4x^{\frac{1}{3}} + 4$.
7. $2\sqrt[3]{x} = \sqrt[6]{x} + 1$.
8. $3x^{-\frac{2}{3}} + 5x^{-\frac{1}{3}} = 2$.
9. $6x^{-1} - x^{-\frac{1}{2}} = 12$.
10. $9x^{-\frac{4}{5}} + 4 = 13x^{-\frac{2}{5}}$.
11. $3\sqrt[4]{x} - 5\sqrt{x} = -2$.
12. $5\sqrt[3]{x^2} = 8\sqrt[3]{x} + 4$.
13. $7\sqrt[5]{x^{-6}} - 4\sqrt[5]{x^{-3}} = 3$.
14. $3\sqrt{2x} - 2\sqrt[4]{2x} = 1$.
15. $(x-1)^2 + 4(x-1) = 21$.
16. $2(x^2-3)^2 - 7(x^2-3) = 30$.
17. $6(x^2+1)^2 + 13(x^2+1) = 28$.
18. $2\sqrt{2x-3} + 5\sqrt[4]{2x-3} = 7$.

19. $(x + 2)^{\frac{1}{2}} - \sqrt[4]{x + 2} = 2.$

20. $(3x - 2)^{\frac{1}{5}} - 4(3x - 2)^{\frac{2}{5}} + 3 = 0.$

21. $3\left(\frac{x}{6} + \frac{2}{x}\right)^2 - 2\left(\frac{x}{6} + \frac{2}{x}\right) = 2\frac{2}{3}.$

22. $3(3x^2 - 2x + 1 - 4\sqrt{3x^2 - 2x + 1} = 15.$

23. $2(2x^2 + 3x - 4)^2 - 3(2x^2 + 3x - 4) = -1.$

24. $x^2 + 7x - 3\sqrt{x^2 + 7x + 1} = 17.$

25. $6(x^2 + x) - 7\sqrt{3x(x + 1) - 2} = 8.$

26. $3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x.$

27. $\sqrt[3]{3x - 2x^2} - (3x - 2x^2)^{\frac{1}{6}} - 2 = 0.$

28. $(x - a)^3 - 3a(x - a)^{\frac{3}{2}} = -2a^2.$

29. $3x^{-\frac{1}{2}} - 7x^{\frac{1}{2}} = 4.$

30. $3x^{\frac{2}{3}} = 8x^{-\frac{2}{3}} - 10.$

31. $16x^{\frac{3}{5}} - 22 = 3x^{-\frac{3}{5}}.$

32. $2x^2 - \sqrt{x^2 - 2x - 3} = 4x + 9.$

33. $5(2x^2 - 1)^{\frac{1}{3}} - 4 = \sqrt[3]{(2x^2 - 1)^{-1}}.$

34. Make up and work an example similar to Ex. 1. To Ex. 12. Ex. 19.

35. Practice oral work with radicals as in Exercise 108 (p. 332).

Radical Equations

231. Radical Equations Resulting in Affected Quadratic Equations. If an equation is cleared of radicals by the methods given in Art. 215 (p. 325), the result is often a quadratic equation.

Ex. Solve $\sqrt{3x + 10} + \sqrt{x + 2} = \sqrt{10x + 16}.$

Squaring, $3x + 10 + 2\sqrt{3x + 10)\,(x + 2)} + x + 2 = 10x + 16$

Hence, $\sqrt{(3x + 10)\,(x + 2)} = 3x + 2$

Squaring again, $3x^2 + 16x + 20 = 9x^2 + 12x + 4$

$$6x^2 - 4x = 16$$

$$x = 2, \quad -\tfrac{4}{3}$$

Substituting these values in the original equation, the only value that verifies is $x = 2$, which is the root. The other value, $x = -\frac{4}{3}$, is not a root of the original equation, but is introduced by squaring in the process of clearing the equation of radical signs. It satisfies the equation,

$$\sqrt{3x+10} - \sqrt{x+2} = \sqrt{10x+16}$$

EXERCISE 116

Solve:

1. $x - 1 - \sqrt{3x-5} = 0.$
2. $2x + 1 - \sqrt{7x+2} = 0.$
3. $x - \sqrt{3x} = 6.$
4. $3x - 2\sqrt{6x} - 6 = 0.$
5. $\sqrt{3x+1} - 2\sqrt{2x+3} = 0.$
6. $2 + \sqrt{2x+7} - \sqrt{5x+4} = 0.$
7. $\sqrt{3x+7} - \sqrt{x+1} - 2\sqrt{x-2} = 0.$
8. $\sqrt{2x+1} - 2\sqrt{x} + \sqrt{x-3} = 0.$
9. $\sqrt{x-a^2} + \sqrt{x+2a^2} - \sqrt{x+7a^2} = 0.$
10. $3\sqrt{x^3+17} - 2\sqrt{5x^3+41} + \sqrt{x^3+1} = 0.$
11. $\sqrt{x+4} + \sqrt{3x+1} - \sqrt{9x+4} = 0.$
12. $2\sqrt{5x} - \sqrt{2x-1} = \dfrac{4x+1}{\sqrt{2x-1}}.$
13. $\dfrac{3\sqrt{2x}-5}{3+\sqrt{2x}} = \dfrac{9-2\sqrt{2x}}{\sqrt{2x}-3}.$
14. $\dfrac{3}{2+\sqrt{12-x}} - \dfrac{\sqrt{12-x}}{5} = 0.$
15. $\dfrac{\sqrt{x}}{\sqrt{x+2}} - \dfrac{\sqrt{x+2}}{\sqrt{x}} - \dfrac{5}{6} = 0.$
16. $\sqrt{x+\frac{2}{3}} + \sqrt{4x-\frac{1}{3}} = \sqrt{5x+\frac{7}{3}}.$
17. $\sqrt{12x^2-x-6} + \sqrt{12x^2+x-6} = \sqrt{24x^2-12}.$
18. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}} - \dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}} = 8\sqrt{x^2-a^2}.$
19. $\sqrt{4x+3} + \sqrt{2x+3} = \sqrt{5x+1} + \sqrt{x+5}.$

20. The square root of a certain number, plus the square root of 1 increased by twice the number, equals 5. Find the number.

21. State Ex. 1 of this Exercise as a problem concerning a number.

22. Similarly state Ex. 3. Ex. 6.

23. Practice oral work with imaginaries as in Exercise 110 (p. 339).

232. **Other Methods of Solving Quadratic Equations,** besides those given in the preceding part of this chapter, may be used. One of these methods may occasionally be used to advantage for some special purpose.

233. **Hindoo Method to Avoid Fractions in Completing the Square.** After simplifying the equation,

Multiply through by four times the coefficient of x^2;

Add to both sides the square of the coefficient of x *in the simplified equation.*

The reason for this process is evident, since if $ax^2 + bx = c$ is multiplied by $4a$, we obtain

$$4a^2x^2 + 4abx = 4ac$$

The addition of b^2 gives on the left-hand side $4a^2x^2 + 4abx + b^2$, which is a perfect square.

Ex. Solve $3x^2 - 2x = 8$ by the Hindoo method.

Multiply by 4×3, or 12,

$$36x^2 - 24x = 96$$

To each member add the square of the coefficient of x in the original equation; that is, add $(-2)^2$, or 4.

$$36x^2 - 24x + 4 = 100$$

Hence,
$$6x - 2 = \pm 10$$
$$6x = 12, \quad -8$$
$$x = 2, \quad -\tfrac{4}{3} \text{ } \textit{Roots}$$

Let the pupil check the work.

EXERCISE 117

Solve:

1. $x^2 + 5x = 6.$
2. $3x^2 - x = 2.$
3. $6x^2 + 5x = 4.$
4. $8x^2 - 2x = 3.$
5. $4x^2 + 4x = 35.$
6. $16x^4 - 40x^2 + 9 = 0.$
7. $6x^2 - ax = 2a^2.$
8. $4a^2x^2 + 5ax = 21.$
9. $(x^2 + 3)^2 - 7(x^2 + 3) = 60.$
10. $4x^{-4} - 101x^{-2} + 25 = 0.$
11. $6\sqrt[3]{x^{-2}} - 11\sqrt[3]{x^{-1}} = 10.$
12. $3(x-2)^2 + 5(x-2) = 12.$
13. $x + \frac{1}{2x} = a + \frac{1}{2a}.$
14. $(a-1)x^2 + (a+1)x = -2.$
15. $(a^2-b^2)x^2 + (a^2+b^2)x = ab.$

234. Use of Formula. Any quadratic equation can be reduced to the form

$$ax^2 + bx + c = 0$$

Solving this equation by use of Art. 226,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By substituting in this result, as a formula, the values of a, b, c in any given equation, the values of x may be obtained.

Ex. Solve $5x^2 + 3x - 2 = 0$ by use of the formula.

Here $a = 5, \quad b = 3, \quad c = -2.$

Substituting for a, b, c in the above formula,

$$x = \frac{-3 \pm \sqrt{9 + 40}}{10} = \frac{-3 \pm 7}{10} = \frac{2}{5}, \quad -1 \textit{ Roots}$$

Let the pupil check the work.

EXERCISE 118

Work the examples in Exercise 117 by use of the formula.

EXERCISE 119

REVIEW

Solve:

1. $6x^2 + x = 1.$

2. $9x^2 - 4 = 2x^4.$

3. $\sqrt{x} - \sqrt[4]{x} = 2.$

4. $x^3 - 16x = 0.$

5. $3\sqrt{x} - 12x^{-\frac{1}{2}} = 5.$

6. $\dfrac{x-1}{x} = 2\frac{1}{6} - \dfrac{x}{x-1}.$

7. $\sqrt{3x} = \dfrac{4}{\sqrt{x-2}} - \sqrt{x-2}.$

8. $\dfrac{x+1}{\sqrt{x}} = \dfrac{a+1}{\sqrt{a}}.$

9. $x + \dfrac{a}{b} = \dfrac{1}{x} + \dfrac{b}{a}.$

10. $\sqrt{4x-3} = 1 + \sqrt{x+1}.$

11. $3x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} = 6.$

12. $x^4 - 27x = 0.$

13. $5x^{-1} + 6x^{-\frac{1}{2}} = 11.$

14. $\dfrac{x-3}{x-4} - \dfrac{x+3}{x+4} = \dfrac{6}{7}.$

15. $\sqrt{5x+4} - \sqrt{3x+1} - \sqrt{x} = 0.$

16. $5(x+2)^4 = 3(x+2)^2 + 2.$

17. $3\sqrt{x+1} - 5\sqrt[4]{x+1} = 2.$

18. $3 - \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right) = \sqrt{x} - \dfrac{1}{\sqrt{x}}.$

19. $20 - 2\left(x + \dfrac{3}{x}\right)^2 = 3\left(x + \dfrac{3}{x}\right).$

20. $\dfrac{3\sqrt{x} - 4\sqrt{2}}{4\sqrt{x} - 2\sqrt{2}} = \dfrac{2\sqrt{3x} - \sqrt{6}}{3\sqrt{3x} - 5\sqrt{6}}.$

21. $4x^2 - 7\sqrt{2x^2 + 3x - 2} = 19 - 6x.$

22. $\dfrac{1}{x}\left\{\dfrac{4x-3}{3} - \dfrac{3x-4}{4}\right\} - \dfrac{2}{3}\left\{\dfrac{x-2}{x+1} - \dfrac{2x-1}{x-1}\right\} + \dfrac{35}{36} = 0.$

23. $\dfrac{abx^2}{a^2 - b^2} + 1 = \dfrac{(a^2 + b^2)x}{a^2 - b^2}.$

24. $\dfrac{8}{\sqrt{x^3}} + \dfrac{7}{\sqrt[4]{x^3}} = 1.$

25. $\dfrac{1}{x + a + b} = \dfrac{1}{x} + \dfrac{1}{a} + \dfrac{1}{b}.$

26. $ax^{2n} + bx^n + c = 0.$

27. $(x^2 + 6x)^2 - 2(x^2 + 6x) = 35.$

28. $45=\frac{n}{2}\left[2+\frac{1}{2}(n-1)\right].$

29. $x^2+(\frac{1}{2}R)^2=R^2.$

30. $3y^2=36+4\sqrt{y^2-7}.$

31. $ax^{\frac{m}{n}}+bx^{\frac{m}{2n}}+c=0.$

32. $y^2+a^2+y=2ay+a+6.$

33. $x^2-1=(1-x)\sqrt{2}-4x.$

34. From $T=2\pi R\,(R+H)$, find R in terms of the other letters.

Find the value of x in the shortest way when

35. $\frac{22}{7}x^2=\frac{22}{7}(225)+\frac{22}{7}(64).$

36. $3.1416x^2=3.1416(441)+3.1416(400).$

37. Who, as far as we know, first solved a quadratic equation, and at about what time? (See p. 462.)

38. How have the different cases in the solution of a quadratic equation been classified at different times?

39. Write (but do not solve) an equation of each of the principal kinds treated in this chapter.

40. Work again Exercise 24 (p. 99).

EXERCISE 120

1. Find two consecutive numbers the sum of whose squares is 61.

2. There are two consecutive numbers such that if the larger be added to the square of the less the sum will be 57. Find the numbers.

3. There are two numbers whose difference is 3, and if twice the square of the larger be added to 3 times the smaller, the sum is 56. Find the numbers.

4. Seven times a certain number is one less than the square of the number next larger than the original number. Find the number.

5. Find the number which increased by its reciprocal equals $\frac{65}{8}$.

6. Find three consecutive numbers such that their sum is 15 less than the square of the smallest.

7. If the length of a rectangle exceeds the width by 5 yd. and the width be denoted by x, express the length and the area in terms of x.

8. The area of a given rectangle is 36 sq. yd., and the length exceeds the width by 15 ft. Find the dimensions of the rectangle.

9. The length of a certain rectangle is twice its width. The rectangle has the same area as another, $1\frac{1}{3}$ times as wide, and shorter by $4\frac{1}{2}$ ft. Find the length of the first rectangle.

10. A rectangular garden contains one-half an acre and the length of the rectangle exceeds its width by 2 rd. Find its dimensions.

11. A square garden contains 100 sq. rd. By how much must its sides be lengthened in order that its area be doubled?

12. A rectangle is 30 × 40 ft. By what per cent must the length and width be increased in order that the area be increased by 528 sq. ft.?

13. A rectangular park is 80 × 100 rd. By adding the same amount to its length and width the area of the park is to be increased by 50%. What is the amount added to each dimension?

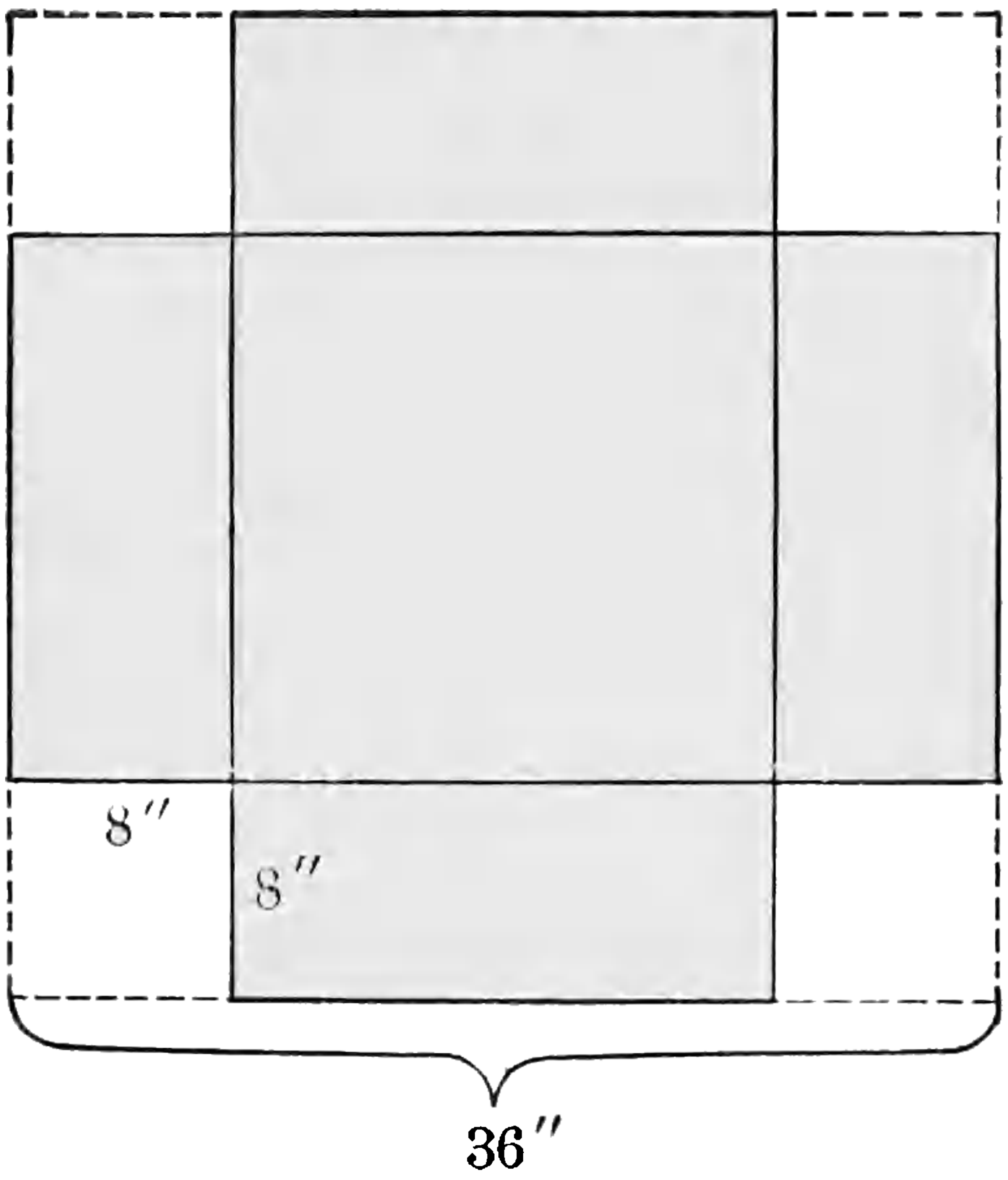
14. A rectangular lot is 8 rd. long and 6 rd. wide, and is surrounded by a drive of uniform width, which occupies $\frac{2}{3}$ as much area as the lot. Required the width of the drive.

15. A farmer has a wheat field 80 rd. long and 60 rd. wide. How wide a strip must be cut around the outside of the field in order to cut 15 A.?

SUG. Draw a diagram of the field with an inner rectangle showing the uncut part left.

If the width of the strip cut is x, what are the dimensions of the inner rectangle? What is the difference between the area of the entire field and that of the inner rectangle?

16. An open box 8 in. deep and to contain 3200 cu. in. is to be formed by cutting out small equal squares from the corners of a square sheet of tin and folding up the sides. Find the length of a side of the square sheet of tin.



17. A number of men bought a yacht costing $2800 and each purchaser paid 7 times as many dollars as there were purchasers. How much did each man pay?

18. One baseball nine has won 5 games out of 13 games played, and another baseball nine has won 9 games out of 15. How many straight games must the first nine win from the second, in order that the average of games won by the two nines shall be the same?

19. One ball nine has won 6 games out of 18, and another has won 12 out of 13. How many straight games will the first team need to win from the second in order that the percentage of games won by the first team shall equal half that of the games won by the second?

20. The numerator of a given fraction exceeds its denominator by 2. Also the given fraction exceeds its reciprocal by $\frac{16}{15}$. Find the fraction.

21. A cistern is filled by two pipes in 18 min.; by the greater alone it can be filled in 15 min. less than by the smaller. Find the time required to fill it by each.

22. A cistern can be filled by 2 pipes in 1 hr. $33\frac{3}{4}$ min., but larger alone can fill it in 1 hr. and 40 min. less than the smaller one. Find the time required by the less.

23. A number of two figures has the units' digit double the tens' digit, but the product of this number and the one obtained by inverting the order of the figures is 1008. Find the number.

24. The left-hand digit of a certain number of two figures is $\frac{3}{2}$ of the right digit. If the product of this number and the number obtained by inverting the order of the digits be increased by twice the original number, the sum is 800. Find the number.

25. A man can row down a stream 16 mi. and back in 10 hr. If the stream runs 3 mi. an hour, find his rate of rowing in calm water.

26. Two trains run at uniform rates over the same 120 mi. of rail; one of them goes 5 mi. an hour faster than the other, and takes 20 min. less time to run this distance. Find the rate of the faster train.

27. One number is $\frac{2}{3}$ of another, and their product, plus their sum, is 69. Find the numbers.

28. Find two numbers whose product is 90 and quotient $2\frac{1}{2}$.

29. Find two numbers whose difference is 4 and the sum of whose squares is 170.

30. Find two consecutive numbers, the difference of whose cubes is 217.

31. If the side of a square is 2 ft., how much must this be increased to increase the area of the square by 153 sq. in.?

32. A farmer has a rectangular wheat field 160 rd. long and 80 rd. wide. How wide a strip must be cut around the outside of the field in order to leave 30 A. uncut?

33. Two workmen can do a piece of work in 24 hours. In how many hours can each do it alone, if it takes one of them 20 hours longer than the other?

34. An open box 6 in. deep and to contain 864 cu. in. is to be formed by cutting out small equal squares from the corners of a square sheet of tin. Find the length of a side of the square sheet of tin.

35. It takes a man 5 hr. to row up a stream 8 mi. and back. If the stream flows at the rate of 3 mi. per hour, what is the rate of the man in still water?

36. A bin is to be constructed to hold 9 T. of coal. If the bin is to be 5 ft. deep and twice as long as it is wide, and if 40 cu. ft. are allowed for 1 T., what will the dimensions of the base of the bin be?

37. The walls and ceiling of a room together contain 104 sq. yd. The room is twice as long as it is wide, and its ceiling is 9 ft. high. Find the length and breadth of the room.

38. If a carriage wheel 11 ft. in circumference took $\frac{1}{12}$ of a second less to revolve, the rate of the carriage would be 1 mi. more per hour. At what rate is the carriage traveling?

39. Find two consecutive numbers the sum of whose squares is a.

40. Find two numbers whose difference is b and the sum of whose squares is c.

41. The area of a given rectangle is p, and the length of the rectangle exceeds the width by q. Find the dimensions of the rectangle.

42. Why are we able to solve many of the problems in the Exercise by algebra and not by arithmetic?

43. If the side of a square is a and an error e is made in measuring the length of one of its sides, what is the error, E, in its area when the area is computed from the side as measured?

44. Make up and work three examples similar to such of the examples in this Exercise as you think are most interesting or instructive.

45. How many examples in Exercise 31 (p. 121) can you now work at sight?

EXERCISE 121

1. In $s = \frac{1}{2}gt^2$, if $g = 32$ ft. and $s = 1000$ ft., find t.

Do you know the meaning of this example as applied to a falling body?

2. In $E = \dfrac{mv^2}{2}$, if $E = 500$, $m = 20$, find v.

3. In $s = \frac{1}{2}gt^2 - vt$, find t, if $s = 200$ ft., $g = 32$ ft., and $v = 60$ ft.

Do you know the meaning of this formula as applied to a body sent upwards from the earth with a velocity of 60 ft. per second?

4. In $h = a + vt - \frac{1}{2}gt^2$, if $h = 500$ ft., $a = 100$ ft., $v = 200$ ft., $g = 32$ ft., find t.

Can you discover the meaning of this example as applied to a body sent upward from the earth from a point 100 ft. above the surface of the earth with a velocity of 200 ft. per second?

5. In $K = \pi R^2$, if $K = 43560$ sq. ft., and $\pi = \frac{22}{7}$, find R.

6. In $T = \pi R(R + L)$, if $T = 220$, $\pi = \frac{22}{7}$, and $L = 20$, find R.

7. In $s = \frac{1}{2}gt^2$, find the value of t in terms of s and g.

8. In $s = vt - 16t^2$, solve for t.

9. In $h = a + vt - 16t^2$, solve for t.

10. In $T = \pi R(R + L)$, find R in terms of T and L.

11. How many seconds will it take a body to fall from rest a distance of 1000 ft. (resistance of air neglected)?

12. If a bullet is fired upward with an average velocity of 2400 ft. per second, how long will it be before the bullet reaches a height of $1\frac{1}{2}$ mi.?

13. If an arrow shot over the top of a steeple reaches the ground in 6 sec. from the time the arrow left the bow, how high is the steeple?

SUG. Use the formula of Ex. 1, letting t = one half of 6.

14. If the steeple of Ex. 13 were 200 ft. high, how many seconds would it be before the arrow returned to the earth?

15. Using the formula $s = \frac{1}{2}gt^2$ (where $g = 32.2$ ft.), find the distance a body will fall from the end of 5.2 sec. to the end of the 7 sec.

16. On the moon, $g = 5.4$ ft. Find the difference in the distance which a body falls on the earth in 5 sec. and the distance it falls on the moon in the same time.

17. Make up and work an example similar to Ex. 12. To Ex. 14.

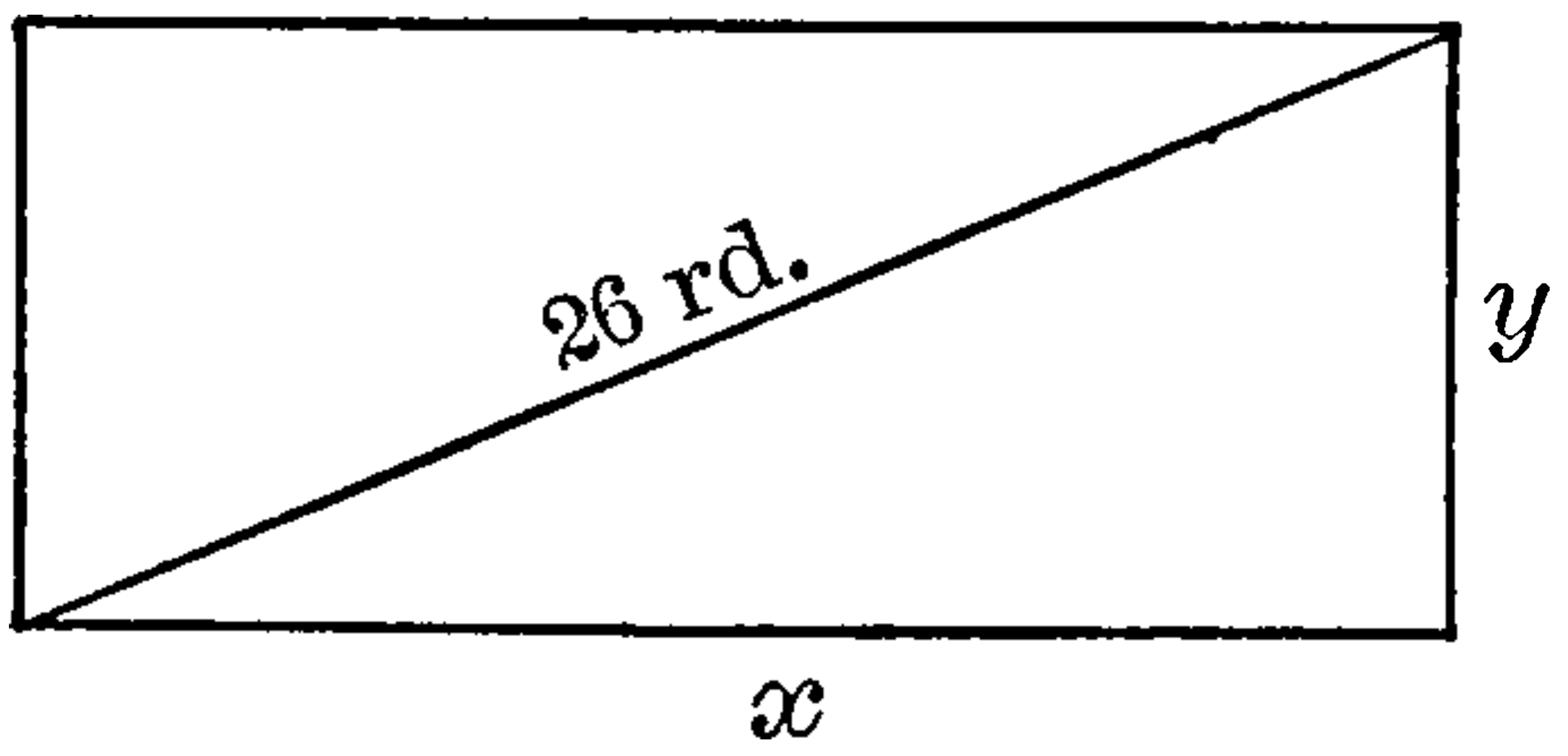
18. How many examples in Exercise 35 (p. 131) can you now work at sight?

CHAPTER XX

SIMULTANEOUS QUADRATIC EQUATIONS

235. Need and Utility of Simultaneous Equations Involving Quadratic Equations.

Ex. A rectangular park is known to contain $1\frac{1}{2}$ acres. The path which leads across it diagonally is measured and found to be 26 rods long. Find the dimensions of the park.



Let x = no. rd. in length of park.
y = no. rd. in width of park.
$1\frac{1}{2}$ acres = 240 sq. rd.
Hence, $x^2 + y^2 = 26^2 \quad \ldots\ldots (1)$
$xy = 240 \quad \ldots\ldots (2)$

By solving (1) and (2) x and y can be determined (see Art. 242).

Try to solve the problem by use of only one unknown, as x. Even if you succeed in getting a solution, you will find the method awkward and inconvenient.

236. Quadratic Equations Containing Two Unknowns. The general quadratic equation containing two unknowns is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

By giving a, b, c, etc., different numerical values including zero, this general equation may be made to take many special forms.

What values must we give a, b, c .... respectively in order to obtain the equation $5x^2 + 3xy + 2y^2 = 5$, from the general equation?

The **absolute term** of an equation is the term which does not contain an unknown factor, as f in the above general equation.

Simultaneous quadratic equations is a brief term for simultaneous equations whose solution involves quadratic equations.

Thus, the equations stated in Art. 235 are simultaneous quadratic equations.

In general, the combination of two simultaneous quadratic equations by elimination gives an equation of the fourth degree in one unknown, which cannot be solved by the methods of this book. Two simultaneous quadratic equations can be solved by elementary methods only in certain special cases.

237. A Homogeneous Equation is one in which all the terms containing an unknown quantity are of the same degree.

Thus, $3x^2y - 5xy^2 + y^3 = 18$ is a homogeneous equation of the third degree. What is the degree of the equation $xy = 6$?

GENERAL METHODS OF SOLUTION

CASE I

238. When One Equation is of the First Degree, the Other of the Second, two simultaneous equations *may always be solved by the method of substitution.*

Ex. Solve
$$\begin{cases} 2x - 3y = 2 & \dots\dots\dots (1) \\ x^2 - 2xy = -7 & \dots\dots\dots (2) \end{cases}$$

Eliminate y, since y occurs only once in equation (2).

From (1),
$$y = \frac{2x-2}{3} \quad \dots\dots\dots (3)$$

Hence from (2), $x^2 - 2x\left(\frac{2x-2}{3}\right) = -7$

Hence,
$$3x^2 - 4x^2 + 4x = -21$$
$$x^2 - 4x = 21$$
$$x = -3,\ 7$$

Substitute for x in (3),
$$y = -\tfrac{8}{3},\ 4$$

$\left.\begin{matrix} x = -3,\ 7 \\ y = -\frac{8}{3},\ 4 \end{matrix}\right\}$ *Roots*

CHECK.

For $x = -3$ and $y = -\frac{8}{3}$

$2x - 3y = -6 + 8 = 2$

$x^2 - 2xy = 9 - 16 = -7$

CHECK.

For $x = 7$ and $y = 4$.

$2x - 3y = 14 - 12 = 2$

$x^2 - 2xy = 49 - 56 = -7$

EXERCISE 122

Find the values of x and y:

1. $3x^2 - 2y^2 = -5.$
$x + y - 3 = 0.$

2. $x - 2y = 3.$
$x^2 + 4y^2 = 17.$

3. $2x^2 + xy = 2.$
$3x + y = 3.$

4. $x^2 - 3y^2 - 1 = 0.$
$x + 2y - 4 = 0.$

5. $x - 3y = 1.$
$7xy - x^2 = 12.$

6. $2x + y + 3 = 0.$
$3x^2 - 7y^2 = 5.$

7. $2x + 5y = 1.$
$2x^2 + 3xy = 9.$

8. $\frac{1}{3}x - \frac{1}{2}y = \frac{1}{3}.$
$(x - y)^2 = y^2 - 7.$

9. $x^2 - 3xy + 2y^2 = 0.$
$2x + 3y = 7.$

10. $x - y = \frac{1}{5}(x + y).$
$x^2 + y^2 = 52.$

11. $9y^2 - 6y - 5 = 3x.$
$9y + x + 5 = 0.$

12. $\frac{3}{y} - \frac{2}{x} = \frac{9}{xy}.$
$\frac{2x}{y} + \frac{10}{xy} - \frac{3y}{x} = 5.$

13. $\frac{2x}{y} = \frac{x + y}{x}.$
$x - y + a = 0.$

14. $x^2 + 2y^2 - 3x = 30.$
$\frac{y}{x} = 2.$

15. $xy = 12.$
$\frac{x}{y} = 3.$

16. $11 = x + 2(y - 1).$
$6 = \frac{y}{2}(x + 1).$

17. $3x - 5y - 1 = 0.$
$2x^2 + 3xy - 5y^2 - 6x + 7y = 4.$

18. $4x^2 - 4xy = y^2 + x + 3y - 1.$
$4x - 2 - 5y = 0.$

19. The sum of two numbers is 5 and the square of the first number increased by twice the square of the second number is 22. Find the numbers.

20. State Ex. 1 as a problem concerning numbers. Ex. 2.

21. How many examples in Exercise 48 (p. 163) can you now work at sight?

CASE II

239. When both Equations are Homogeneous and of the Second Degree, two simultaneous quadratic equations *may always be solved by the substitution* $y = vx$.

Ex. Solve

$$x^2 - xy + y^2 = 21$$
$$y^2 - 2xy = -15$$

Substitute $y = vx$, $x^2 - vx^2 + v^2x^2 = 21 \quad \ldots \quad (1)$

$$v^2x^2 - 2vx^2 = -15 \quad \ldots \quad (2)$$

From (1), $x^2 = \dfrac{21}{1 - v + v^2} \quad \ldots \quad (3)$

From (2), $x^2 = \dfrac{-15}{v^2 - 2v} \quad \ldots \quad (4)$

Equate the values of x^2 in (3) and (4),

$$\frac{21}{1 - v + v^2} = \frac{-15}{v^2 - 2v}$$

Hence,

$$21v^2 - 42v = -15 + 15v - 15v^2$$
$$36v^2 - 57v = -15$$
$$12v^2 - 19v = -5 \qquad \therefore v = \tfrac{5}{4}, \tfrac{1}{3}$$

If $v = \dfrac{5}{4}$, $x^2 = \dfrac{21}{1 - \frac{5}{4} + \frac{25}{16}}$

$x = \pm 4$

$y = vx = \frac{5}{4}(\pm 4)\ \pm 5$

If $v = \dfrac{1}{3}$, $x^2 = \dfrac{21}{1 - \frac{1}{3} + \frac{1}{9}}$

$x = \pm 3\sqrt{3}$

$y = vx = \frac{1}{3}(\pm 3\sqrt{3}) = \pm\sqrt{3}$

Hence,

$$\left.\begin{array}{l} x = \pm 4, \ \pm\sqrt{3} \\ y = \pm 5, \ \pm\sqrt{3} \end{array}\right\} \textit{Roots}$$

Let the pupil check the work.

Two simultaneous equations of the kind treated in Case II may also be solved by eliminating the absolute term between them and factoring to find the value of one unknown in terms of the other and then proceeding as in Case I.

EXERCISE 123

Find the values of x and y:

1. $x^2 + 3xy = 28.$
$xy + 4y^2 = 8.$

2. $2x^2 + xy = 15.$
$x^2 - y^2 = 8.$

3. $x^2 + 3xy = 7.$
$y^2 + xy = 6.$

4. $2x^2 = 46 + y^2.$
$xy + y^2 = 14.$

5. $3x^2 + y^2 = 12.$
$5xy - 4x^2 = 11.$

6. $2y^2 - 4xy + 3x^2 = 17.$
$y^2 - x^2 = 16.$

7. $x^2 + xy + 2y^2 = 74.$
$2x^2 + 2xy + y^2 = 73.$

8. $2x^2 + 3xy + y^2 = 14.$
$3x^2 + 2xy - 4y^2 = 9.$

9. $4xy - x^2 = 5.$
$13x^2 - 31xy + 16y^2 = 2\frac{1}{2}.$

10. $x^2 + xy + 2y^2 = 44.$
$2x^2 - xy + y^2 = 16.$

Also solve the following miscellaneous examples:

11. $3y - 1 = x.$
$5y^2 - x^2 = 1.$

12. $2x^2 - 3y^2 = 6.$
$3xy - 4y^2 = 2.$

13. $x + y + 3xy = 83.$
$3x - y - 1 = 0.$

14. $10x + y - (10y + x) = 9.$
$(10x + y)(10y + x) = 736.$

15. $x - y = 0.$
$5xy + y^2 = 54.$

16. $x^2 = 5 + 2xy.$
$x^2 + y^2 = 29.$

17. Point out the examples in Exercise 128 (p. 380) which come under Case I. Under Case II.

18. Point out the homogeneous equations in Exs. 10–15 of Exercise 122.

19. Express Ex. 1 above as a problem concerning two numbers.

20. Work the example of Art. 235 (p. 366) by the method of Art. 239.

21. Find a number consisting of two digits such that if the number is multiplied by the left-hand digit, the result will be 260. But if the number is multiplied by the right-hand digit, the result will be 104.

22. Make up (but do not solve) an example in each of the cases studied thus far in this chapter.

23. How many examples in Exercise 51 (p. 172) can you now work at sight?

SPECIAL METHODS OF SOLVING SIMULTANEOUS QUADRATICS

240. The methods of Cases I and II are the only general methods which can be used in solving all simultaneous quadratic equations of a given class. Besides these, however, there are certain special methods which enable us to solve important particular examples.

Examples which come directly under Cases I and II are often solved more advantageously by one of these special methods.

The special methods apply with particular advantage to symmetrical equations.

241. A Symmetrical Equation is one in which, if y is substituted for x, and x for y, the resulting equation is identical with the original equation.

Thus, each of the following is a symmetrical equation:

$$x^3 + 3x^2y^2 + y^3 = 18$$
$$x + y = 12$$
$$xy = 6$$

CASE III

242. Addition and Subtraction Method (often in connection with multiplication and division). In this method the object is to find *first, the values of* $x + y$ *and* $x - y$*, and then the values of* x *and* y *themselves.*

Ex. 1. Solve
$$\begin{cases} x + y = 7 & \dots\dots\dots (1) \\ xy = 12 & \dots\dots\dots (2) \end{cases}$$

Here we have the value of $x + y$ given, and the first object is to find the value of $x - y$.

Square (1), $x^2 + 2xy + y^2 = 49$ . . . . (3)

Multiply (2) by 4, $4xy = 48$ . . . . (4)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 1$ . . . . (5)

Extract square root of (5), $x - y = \pm 1$ . . . . . . (6)

Add (1) and (6), divide by 2, $x = 4 \text{ or } 3$

Subtract (6) from (1), divide by 2, $y = 3 \text{ or } 4$ } *Roots*

Let the pupil check the work.

Ex. 2. Solve
$$\begin{cases} x^3 + y^3 = 65 & \dots\dots\dots (1) \\ x + y = 5 & \dots\dots\dots (2) \end{cases}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13$ . . . . (3)

Square (2), $x^2 + 2xy + y^2 = 25$ . . . . (4)

Subtract (3) from (4), $3xy = 12$

Hence, $xy = 4$ . . . . (5)

Subtract (5) from (3), $x^2 - 2xy + y^2 = 9$

$\therefore x - y = \pm 3$

But $x + y = 5$

Hence,
$$\left.\begin{aligned} x &= 4, 1 \\ y &= 1, 4 \end{aligned}\right\} \textit{Roots}$$

Ex. 3. Solve
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 11 & \dots\dots\dots\dots (1) \\[2ex] \dfrac{1}{x^2} + \dfrac{1}{y^2} = 61 & \dots\dots\dots\dots (2) \end{cases}$$

Squaring (1), $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = 121$ . . . . . . (3)

Subtracting (2) from (3), $\dfrac{2}{xy} = 60$ . . . . (4)

Subtracting (4) from (2), $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; (5)$

Hence, $\frac{1}{x} - \frac{1}{y} = \pm 1$

But, from (1), $\frac{1}{x} + \frac{1}{y} = 11$

Hence, adding, $\frac{2}{x} = 12, \ 10$

$$\left.\begin{aligned} \therefore x &= \tfrac{1}{6}, \ \tfrac{1}{5} \\ y &= \tfrac{1}{5}, \ \tfrac{1}{6} \end{aligned}\right\} \textit{Roots}$$

Let the pupil check the work.

EXERCISE 124

Find the values of x and y:

1. $x + y = 13.$
$xy = 36.$

2. $x^2 + y^2 = 25.$
$x + y = 1.$

3. $x + y = -10.$
$xy = 21.$

4. $x^2 + xy + y^2 = 21.$
$x + y = -1.$

5. $x^2 - xy + y^2 = 37.$
$x^2 + xy + y^2 = 79.$

6. $x^2 + y^2 = 2\frac{1}{2}.$
$3xy = 2\frac{1}{4}.$

7. $x + y + 1 = 0.$
$xy + 3\frac{1}{9} = 0.$

8. $x^3 + y^3 = 9.$
$x + y = 3.$

9. $x^3 + y^3 = 37.$
$x + y = 1.$

10. $x^3 + y^3 - 218 = 0.$
$x^2 - xy + y^2 = 109.$

11. $x^2 + 3xy + y^2 = -2\frac{3}{4}.$
$x^2 - xy + y^2 = 12\frac{1}{4}.$

12. $xy - 6a^2 = 0.$
$x^2 + y^2 = xy + 7a^2.$

13. $x^3 + y^3 = 2a^3 + 6a.$
$x^2 - xy + y^2 = a^2 + 3.$

14. $\frac{x}{y} + \frac{y}{x} = 2\frac{1}{6}.$
$x + y = 5.$

15. $x^3 + y^3 = 224.$
$x^2y + xy^2 = 96.$

16. $\frac{1}{x^2} + \frac{1}{y^2} = 13.$
$\frac{1}{xy} - 6 = 0.$

17. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=3\frac{1}{2}.$
$\dfrac{1}{x}+\dfrac{1}{y}=2.$

18. $x^3+y^3=-\frac{7}{3}xy.$
$x+y=\frac{2}{3}.$

19. $x^4+x^2y^2+y^4=4\frac{4}{27}.$
$x^2+xy+y^2=1\frac{1}{3}.$

20. $x^2+y^2=\frac{17}{2}.$
$x-y=4.$

21. $x^3-y^3=98.$
$x-y=2.$

22. $x^2+y^2=5(a^2+b^2).$
$y-x=a+3b.$

23. $3x^2+5xy+3y^2=13.$
$5x^2+3xy+5y^2=27.$

Solve the following miscellaneous examples:

24. $x=y+5.$
$\dfrac{1}{x}+\dfrac{1}{y}=-\dfrac{1}{6}.$

25. $x^2+xy=6.$
$6y^2=8-xy.$

26. $x^2+y^2=5.$
$x+y=3.$

27. $\dfrac{y+x}{y}=3.$
$x^2-2y^2=18.$

28. $x^3-y^3=7.$
$x-y=1.$

29. $\frac{1}{2}(x-y)=x-4.$
$xy=2x+y+2.$

30. $x+y=2a.$
$x^2+y^2=2a^2.$

31. Find two numbers such that their sum is 14 and their product is 48.

32. State Ex. 1 as a problem concerning two unknown numbers. In like manner state Ex. 2. Ex. 8. Ex. 17.

33. Make up and solve a problem concerning two unknown numbers such that the solution involves quadratic equations.

34. Point out the symmetrical equations in the examples in this Exercise.

35. Make up (but do not solve) an Example in Case I; one in Case II; and five different examples in Case III.

36. Practice oral work with fractions as in Exercise 58 (p. 190).

CASE IV

243. Solution by the Substitutions, $x = u + v$ and $y = u - v$.

Ex. Solve $\begin{cases} x^5 + y^5 = 242 \ldots\ldots (1) \\ x + y = 2 \ldots\ldots (2) \end{cases}$

Substitute $x = a + b$, $y = a - b$ in (1) and (2),

Then, $(a + b)^5 + (a - b)^5 = 242 \ldots\ldots (3)$

$a + b + a - b = 2 \ldots\ldots (4)$

From (3), $2a^5 + 20a^3b^2 + 10ab^4 = 242 \ldots\ldots (5)$

From (4), $2a = 2 \ldots\ldots (6)$

Divide (5) and (6) by 2, and substitute 1 for a in (5)

$$1 + 10b^2 + 5b^4 = 121$$

Hence, $b = \pm 2, \ \pm \sqrt{-6}$

But $a = 1$

Hence, $\left.\begin{array}{l} x = a + b = 3, -1, 1 \pm \sqrt{-6} \\ y = a - b = -1, 3, 1 \mp \sqrt{-6} \end{array}\right\}$ *Roots*

Let the pupil check the work.

EXERCISE 125

Solve:

1. $x^5 + y^5 = 244.$
   $x + y = 4.$
2. $x^4 + y^4 = 82.$
   $x + y = 2.$
3. $x^5 + y^5 = 211.$
   $x + y = 1.$
4. $x^4 + y^4 = 257.$
   $x - y = 5.$
5. $x^5 - y^5 = 2.$
   $x - y = 2.$
6. $x^4 + 16y^4 = 97.$
   $x + 2y = 5.$

Work the following miscellaneous problems:

7. $x + 4y = 14.$
   $y^2 + 4x = 2y + 11.$
8. $x^2 = y^2 + 3.$
   $5x^2 + 3y^2 = 15 + 4xy.$
9. $\frac{x}{y} + \frac{y}{x} = \frac{17}{4}.$
   $x - 2y = 2.$
10. $x^2 + xy + y^2 = 7.$
    $x^4 + x^2y^2 + y^4 = 91.$

11. Solve Ex. 1. by dividing the first equation by the second.

12. Make up (but do not solve) an example illustrating each of the cases studied thus far in this chapter.

CASE V

244. Use of Compound Unknown Quantities. It is often expedient to *consider some expression (as the sum, difference, or product of the unknown quantities) as a single unknown quantity, and find its value, and hence the value of the unknown quantities themselves.*

Ex. Solve
$$\begin{cases} x^2+y^2=18-x-y \quad \dots\dots (1) \\ xy=6 \quad \dots\dots\dots\dots (2) \end{cases}$$

Multiply (2) by 2 $\quad 2xy=12 \quad \dots\dots\dots (3)$

Add (3) and (1) $\quad x^2+2xy+y^2=30-x-y \quad \dots\dots (4)$

Let $\quad x+y=v$

Then from (4),
$$\begin{aligned} v^2&=30-v \\ v^2+v&=30 \\ v&=-6,\ 5 \end{aligned}$$

Hence,
$$\begin{aligned} x+y&=-6 \\ xy&=6 \\ \therefore x&=-3\pm\sqrt{3} \\ y&=-3\mp\sqrt{3} \end{aligned}\Bigg\}\ \textit{Roots}$$

also
$$\begin{aligned} x+y&=5 \\ xy&=6 \\ \therefore x&=3,\ 2 \\ y&=2,\ 3 \end{aligned}\Bigg\}\ \textit{Roots}$$

Let the pupil check the work.

EXERCISE 126

Solve and check:

1. $x+y+\sqrt{x+y}=6.$
$xy=3.$

2. $x^2y^2+xy=6.$
$x+2y=-5.$

3. $xy+\sqrt{xy}=6.$
$x+y=5.$

4. $\sqrt{\frac{x}{y}} + \frac{x}{y} = 6.$
$x - 2y = 2.$

5. $x - y + \sqrt{x - y} = 6.$
$xy = 5.$

6. $x^2 + y^2 + x + y = 24;\ xy = -12.$
$x = 3, -4, \pm 2\sqrt{3};\ y = -4, 3, \mp 2\sqrt{3}.$

7. $x^2 + y^2 = x - y + 50.$
$xy = 24.$

8. $x^2y^2 + 7xy = -6.$
$5x^2 + xy = 4.$

9. $(x - y)^2 - 3(x - y) = 40.$
$x^2y^2 - 3xy = 54.$

10. $x^2 + y^2 + x + 5y = 6.$
$xy - 2y = -2.$

11. $x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) = 70.$
$x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}} + y^{\frac{1}{2}} = 17.$

Also solve the following miscellaneous examples:

12. $\frac{y - x}{x} = 1.$
$xy = 8.$

13. $xy + 3y^2 - 20 = 0.$
$x^2 - 3xy + 8 = 0.$

14. $x^2y^2 + 3xy = 18.$
$x + 2y = 5.$

15. $\frac{x^2}{y} + \frac{y^2}{x} = -\frac{7}{2}.$
$x + y = 1.$

16. $x + y + \sqrt{x + y} = 12.$
$xy = 20.$

17. $x^2 + y^2 = 125.$
$xy = 22.$

18. $\frac{x + 2}{y - 1} = 3.$
$2x^2 - xy = 20.$

19. $\frac{x}{a} + \frac{y}{b} = 1.$
$\frac{a}{x} + \frac{b}{y} = 4.$

20. Make up (but do not solve) an example illustrating each of the cases studied thus far in the chapter.

CASE VI

245. Factorial Solutions. The solution of a set of simultaneous quadratic equations is often facilitated by the use of factoring.

Thus, the solution of Ex. 9 of Exercise 122 (p. 368) may be shortened by factoring in the first equation. The two following equations will then be obtained:

$$\begin{cases} (x-y)(x-2y)=0 \\ 2x+3y=7 \end{cases}$$

Since the first equation is satisfied either when $x - y = 0$ or $x - 2y = 0$, we obtain the following two sets of equations as equivalent to the original single set or system:

$$\begin{cases} x-y=0 \\ 2x+3y=7 \end{cases}$$

Whence, $\left.\begin{aligned} x &= \tfrac{7}{5} \\ y &= \tfrac{7}{5} \end{aligned}\right\}$ *Ans.*

$$\begin{cases} x-2y=0 \\ 2x+3y=7 \end{cases}$$

Whence, $\left.\begin{aligned} x &= 2 \\ y &= 1 \end{aligned}\right\}$ *Ans.*

Let the pupil check the work.

The solution of this example by the factorial method requires less than one fourth the labor involved in the solution by the method of Art. 238.

In general, if A, B, C, and D are algebraic expressions integral with reference to x and y, and if

$$\begin{cases} A \cdot B \cdot C = 0 \\ D = 0 \end{cases}$$

the given set of equations is equivalent to the following three sets:

$$\begin{cases} A = 0 \\ D = 0 \end{cases} \qquad \begin{cases} B = 0 \\ D = 0 \end{cases} \qquad \begin{cases} C = 0 \\ D = 0 \end{cases}$$

Hence, let the pupil state the sets of simple equations whose solution is equivalent to the solution of

$$\begin{cases} (x-2y)(3x+y)(x-3y)=0 \\ 5x-y=0 \end{cases}$$

246. Factorial Method of Solution Aided by Division. If A, B, C, and D have the meaning given in Art. 245, and if

$$\begin{cases} A \cdot C = B \cdot D \\ A = B \end{cases}$$

it is evident that this system of equations is satisfied either when

$$\begin{cases} C = D \\ A = B \end{cases} \quad \text{or} \quad \begin{cases} A = 0 \\ B = 0 \end{cases}$$

Hence, the last two sets form a system equivalent to the original system.

Note that the equation $C = D$ is obtained by dividing the members of $A \cdot C = B \cdot D$ by the corresponding members of $A = B$.

Ex. Solve

$$x^3 + x = 9y^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$x^2 + 1 = 6y \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Writing equation (1) in the form $x(x^2 + 1) = 6y(\frac{3}{2}y)$, we obtain the two systems which follow:

$$x = \tfrac{3}{2}y$$

$$x^2 + 1 = 6y$$

$$\text{Whence, } \left.\begin{aligned} x &= 2 \pm \sqrt{3} \\ y &= \tfrac{2}{3}(2 \pm \sqrt{3}) \end{aligned}\right\} \textit{Roots}$$

$$x^2 + 1 = 0$$

$$6y = 0$$

$$\text{Whence, } \left.\begin{aligned} x &= \pm \sqrt{-1} \\ y &= 0 \end{aligned}\right\} \textit{Roots}$$

Let the pupil check the work.

EXERCISE 127

Solve by the factorial method:

1. $x - 2y = 4.$
   $x^2 - 5xy + 4y^2 = 0.$

2. $x - 3y = 1.$
   $x^2 + 3xy + 2y^2 = 0.$

3. $x(x - y) = 0.$
   $x^2 - 3xy - 4y^2 = 0.$

4. $(x - 2)(y - 2) = 0.$
   $3x^2 - 4xy + y^2 = 0.$

5. $(x - 2)(y + 3) = 0.$
   $(x - 2y)(x + 2y) = 0.$

6. $1 + x^3 = y^3.$
   $1 + x = y.$

7. $y(x + 3) = 9x^2 - 1.$
   $y = 3x - 1.$

8. $y^3 + y = 9x^2.$
   $y^2 + 1 = 6x.$

9. Practice the oral solution of simple equations as in Exercise 64 (p. 209).

10. Solve Ex. 1 of Exercise 123 (p. 370) by eliminating the absolute term in the two equations and applying the factorial method to the resulting equation.

11. In the same way work Ex. 2 of Exercise 123 (p. 370). Also Ex. 3.

EXERCISE 128

REVIEW

In solving a system of simultaneous quadratic equations, the first thing to notice is the degree of each equation.

Find the values of x and y:

1. $2x - 5y = 0.$
$x^2 - 3y^2 = 13.$

2. $x + y = 2.$
$\frac{2}{x} + \frac{3}{y} = 6.$

3. $2x^2 - xy = 28.$
$x^2 + 2y^2 = 18.$

4. $x - y + \sqrt{x - y} = 12.$
$x + y = 11.$

5. $x^3 + y^3 = 91.$
$x + y = 1.$

6. $\frac{1}{x} + \frac{1}{y} = 13.$
$\frac{1}{xy} = 36.$

7. $x^2 + 3y^2 = 28.$
$x^2 + xy + 2y^2 = 16.$

8. $\frac{2x}{y} = \frac{x + y}{x}.$
$x - y + a = 0.$

9. $(x - 1)(y - 2) = 0.$
$(2x - 5y)(3x - y + 1) = 0.$

10. $x^4 + y^4 = 17.$
$x + y = 3.$

11. $xy + 2x = 5.$
$2xy - y = 3.$

12. $x^2 + 2x - y = 5.$
$2x^2 - 3x + 2y = 8.$

13. $(x + 2)(2y - 1) = 35.$
$xy - x - y = 7.$

14. $x^2y^2 - 5xy + 6 = 0.$
$5x + 3y = 14.$

15. $x^3 - y^3 = -3xy.$
$x - y = 2.$

16. $(x + y)^2 - (x + y) = 20.$
$2x^2 - 3x + 4y = 14.$

17. $\frac{x^2}{y} + \frac{y^2}{x} = \frac{3}{2}.$
$\frac{1}{x} + \frac{1}{y} = 4\frac{1}{2}.$

18. $x^2 + y^2 + x + 3y = 18.$
$xy - y = 12.$

19. $x^{-1} + y^{-1} = 2.$
$x^{-2} + y^{-2} = 2\frac{1}{2}.$

20. $x^{-2} + y^{-2} = 13.$
$6xy = 1.$

21. $x^2 + y^2 = 13(b^2 + 1).$
$x + y = 5b - 1.$

22. $x^2 + y^2 + x + y = 14.$
$xy + x + y = -5.$

23. $3x^2 - 35 = 5xy - 7y^2.$
$2x^2 - 35 = y^2 - xy.$

24. $a + ar + ar^2 = 65.$
$a + ar^2 = \frac{10}{3}ar.$

25. $11 = a + 2(n - 1).$
$36 = \frac{n}{2}(a + 11).$

26. $x^{\frac{1}{3}} - y^{\frac{1}{3}} = 2.$
$x - y = 26.$

27. $\frac{1}{x} - \frac{2}{y} = 3.$
$\frac{1}{x^2} + \frac{4}{y^2} = 17.$

28. $x + y = 65.$
$\sqrt[3]{x} + \sqrt[3]{y} = 5.$

29. $x + \sqrt{xy} + y = 14.$
$x^2 + xy + y^2 = 84.$

30. $x^2 + 4y^2 + 80 = 15x + 30y.$
$xy - 6 = 0.$

31. $x - y = \sqrt{x} + \sqrt{y}.$
$x^{\frac{3}{2}} - y^{\frac{3}{2}} = 37.$

32. $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 4.$
$x + y = 28.$

33. $x^4 + y^4 = 17.$
$xy = 2.$

34. $x^2 + y^2 = xy + 7.$
$x - y = xy - 5.$

35. $x^2 = 4(a^2 + b^2 - y^2).$
$xy = 2ab.$

36. $x^{-1} + y^{-1} = 5.$
$(x + 1)^{-1} + (y + 1)^{-1} = \frac{17}{12}.$

37. $x - 4 = y(x - 2).$
$y - 8 = x(y - 2).$

38. $2x^{-1} + 5y^{-1} = 4.$
$x^{-2} - 2x^{-1}y^{-1} + y^{-2} = 1.$

39. $xy + x + y = 5.$
$x^2y + xy^2 = -84.$

40. $ax + by = 0.$
$(ax - 2)(by + 3) = -2.$

41. $x^3 + x = 9y.$
$x^2 + 1 = 6y.$

42. $x^2 + y^2 = 3xy - 4.$
$x^4 + y^4 = 272.$

43. $a(x - a) = b(y - b).$
$xy = ax + by.$

44. $x^4 + a^4y^4 = \frac{17}{4}a^2x^2y^2.$
$3x + ay = 5.$

45. $xy = a^2.$
$xz = b^2.$
$yz = c^2.$

46. $xy + xz = a.$
$xy + yz = b.$
$xz + yz = c.$

47. $\frac{xy}{x + y} = \frac{2}{3}.$
$\frac{xz}{x + z} = \frac{3}{4}.$
$\frac{yz}{y + z} = \frac{6}{5}.$

48. Make up (but do not solve) an example of each of the cases and principal sub-forms in each case, treated in this chapter.

EXERCISE 129

1. The sum of the squares of two numbers is 58, and their product is 21. Find the numbers.

2. Separate 32 into two parts such that their product shall be 112.

3. Two numbers when added produce 5.7, and when multiplied produce 8. Find the numbers.

4. What are the two parts of 18 whose product exceeds 8 times their difference by 1?

5. The sum of two numbers increased by three times their product is 83; also three times the less number exceeds the larger number by 1. Find the numbers.

In working the following examples concerning rectangles, *draw a diagram* for each rectangle considered.

6. The area of a rectangle is 84 sq. ft. and the distance around it (perimeter) is 38 ft. Find the length and breadth (dimensions) of the rectangle.

7. The diagonal of a rectangle is $\sqrt{\frac{5}{2}}$. If each side of the rectangle were increased by 1, the area would be increased by 3. What are the sides?

8. The area of a rectangular garden is 1200 sq. yd. If the width were increased by 5 yd. and the length by 10 yd., the area would be 1750 sq. yd. Find the dimensions of the rectangle.

9. The area of a (double) tennis court is 312 sq. yd., and the perimeter is 76 yd. Find the dimensions of the court in feet.

10. If the dimensions of a rectangular field were each increased by 3 rd., its area would be 140 sq. rd.; but if its width were increased by 8 rd. and its length diminished by 2, its area would be 135 sq. rd. Find its actual dimensions.

11. A rectangular lot containing 270 sq. rd. is surrounded by a road 1 rd. wide; the area of the road is 70 sq. rd. Find the dimensions of the field.

12. A hall of 90 sq. yd. can be paved with 720 rectangular tiles of a certain size, but if each tile were 3 in. shorter and 3 in. wider, it would require 648 tiles. What is the size of each tile?

13. The area of a given rectangle is 800 sq. ft. If the length and width of the rectangle be each increased by 20%, the area will be increased by 44%. Find the dimensions of the rectangle.

14. If a train had traveled 6 miles an hour faster, it would have required 1 hour less to run 180 miles. How fast did it travel?

SUG. Let x = the number of miles the train travels per hour at first, and y = the number of hours it travels. Then what will represent the number of miles per hour and the number of hours at the second rate?

15. A gentleman distributed $9 equally among some boys. If he had begun by giving each boy 5 cents more, 6 of them would have received nothing. How many boys were there?

16. A number of men agreed to buy a boat for $7200, but 3 of their number died, and each survivor was obliged to contribute $400 more than he otherwise would have done. How many men were there?

17. A certain club owes a debt of $400, but is informed by the treasuer that if 5 new members are admitted, the assessment to meet the debt will be \$4 less per member. How many members has the club?

18. The price of photographs is raised \$3 per dozen, and customers consequently receive 10 photographs less than before for \$5. Find the old and new price for a single photograph.

19. A certain number of eggs cost a dollar, but if there had been 10 more eggs at the same price, they would have cost 6¢ a dozen less. What was the price of a dozen eggs?

20. A given fraction when reduced to its lowest terms equals $\frac{3}{4}$. Also if 3 is subtracted from the numerator of the fraction, the fraction is the same as if 6 had been added to its denominator. Find the fraction.

21. The numerator of a given improper fraction exceeds its denominator by 1. Also the given fraction exceeds its reciprocal by $\frac{7}{12}$. Find the fraction.

22. The sum of the numerator and denominator of a certain fraction is 8, and if $2\frac{1}{2}$ be added to each term of the fraction, its value will be increased by $\frac{2}{15}$. What is the fraction?

23. A baseball nine has won $\frac{2}{3}$ of the games played. If it should play 16 more games and win half of them, its average of games won would be $\frac{4}{5}$ of what it would be if it should play 8 more games and win all of them. How many games has it played, and how many has it won?

24. A certain number of two figures, when multiplied by the left digit, becomes 56; but when multiplied by the right digit, it becomes 224. Find the number.

25. Make up and work an example similar to Ex. 24.

26. A man finds that he can row 12 miles down stream in 2 hours, but that it takes him 4 hours to row 6 miles down stream and back. Find his rate in still water and the rate of the stream.

27. A crew rowing at $\frac{2}{3}$ their usual rate took 32 hours to row down stream 48 miles and back to starting-place. Had they rowed at their usual rate it would have taken 18 hours for the same circuit. Find their rate and that of the stream.

28. Two trains traveling toward each other left, at the same time, two stations 240 miles apart. Each reached the station from which the other started, the one $3\frac{3}{5}$ hours, and the other $1\frac{3}{5}$ hours, after they met. Find their rates of running.

29. The difference of two numbers is 5, and the difference of their cubes is 215. Find the numbers.

30. Divide the number 12 into two parts such that the sum of the fractions obtained by dividing 12 by the parts shall be $\frac{64}{15}$.

31. Find two numbers whose product is 42, such that if the larger be divided by the less, the quotient is 4 and the remainder 2.

32. In placing telephone poles between two places, it was found that if the poles were placed 10 ft. further apart than was originally planned, 4 poles less per mile were needed. How far apart were the poles placed at first?

33. A girl has 12,000 words to write. If she uses a typewriter she can write 25 words more per minute than she can with the pen, and it will take $8\frac{1}{3}$ hours less to write the 12,000 words. What is her rate per minute with the pen?

34. Two square plots contain together 610 sq. ft., but a third plot, which is 1 ft. shorter than a side of the larger square, and 1 ft. wider than the less, contains 280 sq. ft. What are the sides of the two squares?

35. Find two fractions whose sum is equal to their product, and the difference of whose squares is $\frac{5}{6}$ of their product.

36. A man finds that he can row 8 miles up stream in 4 hours, but that he can row 8 miles down stream and back in 5 hours. Find his rate in still water and also the rate of the stream.

37. The area of a given rectangle is 2400. If its length and width were each increased by 50%, the area of the rectangle would be increased by 125%. Find the dimensions of the rectangle.

38. The hypotenuse of a right triangle is 20 and the sum of the other two sides is 28. Find the length of the sides.

39. The fore wheel of a carriage makes 28 revolutions more than the hind wheel in going 560 yd., but if the circumference of each wheel were increased by 2 ft., the difference would be only 20 revolutions. What is the circumference of each wheel?

40. Find two numbers such that their sum is a and their product b.

41. Why are we able to solve many of the problems in this exercise by algebra and not by arithmetic?

42. Make up and work an example similar to Ex. 6. To Ex. 17. Ex. 24.

CHAPTER XXI

GRAPHS OF QUADRATIC AND HIGHER EQUATIONS

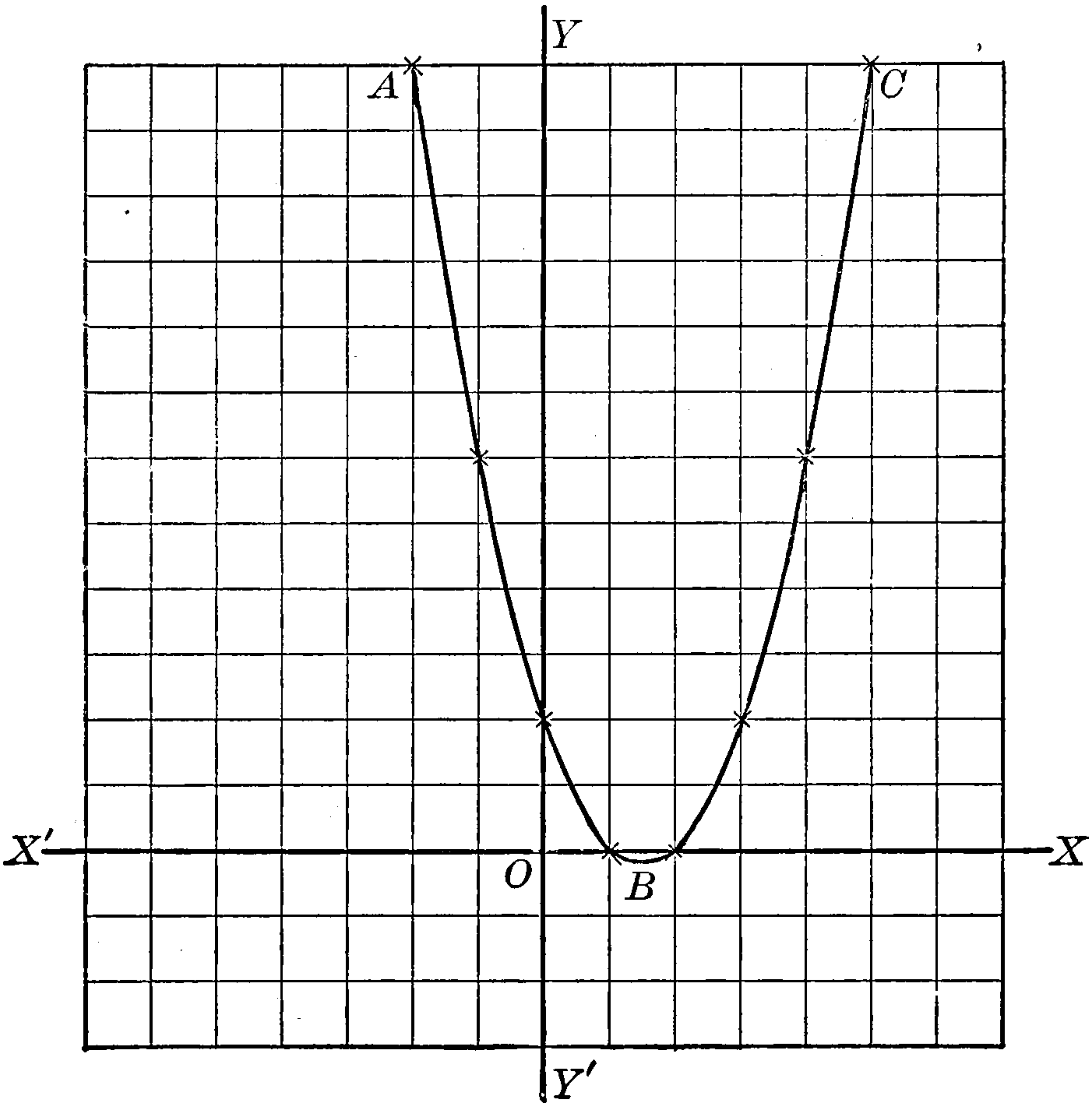
247. Graph of a Quadratic Equation of Two Unknown Quantities.

Ex. 1. Construct the graph of $y = x^2 - 3x + 2$.

x	y
0	2
1	0
2	0
3	2
$\frac{3}{2}$	$-\frac{1}{4}$
	etc.

x	y
-1	-6
-2	12
etc.	

The graph obtained is the curve *ABC*. A curve of this kind is called a *parabola*. The path of a projectile, for instance that of a baseball when thrown or batted (resistance of the air being neglected), is an arc of a parabola.



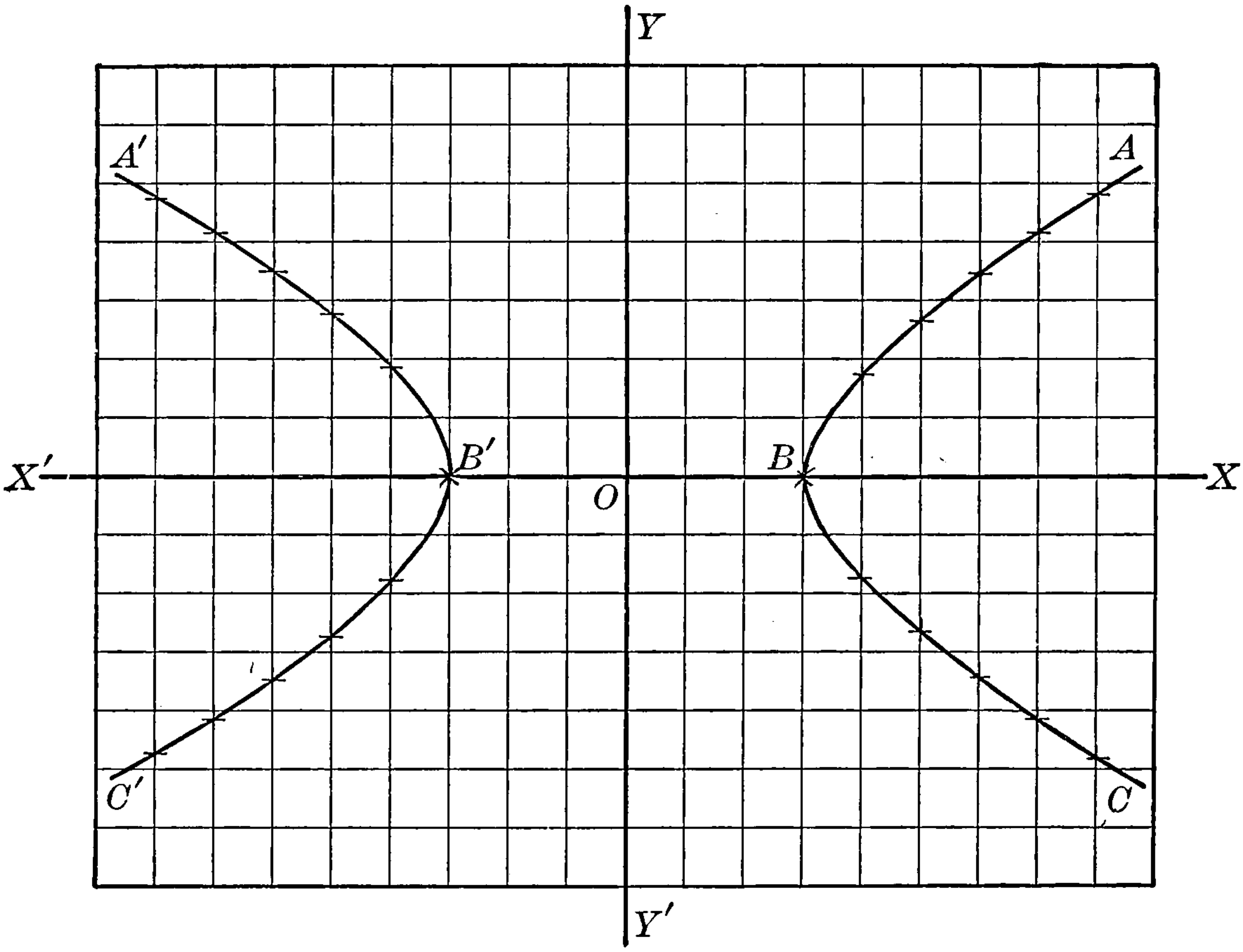
It will be noted that the above method of graphing is the same as that given in Art. 148 (p. 255), but that here it is sometimes advantageous to let x have fractional values as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{10}$, $\frac{3}{2}$, etc. The observant pupil will also find methods of abbreviating the work in certain cases.

In general, it will be found that the graph of a quadratic equation of two unknown quantities is a curved line, and, in particular, either a circle, parabola, ellipse, or hyperbola.

Ex. 2. Construct the graph of $4x^2 - 9y^2 = 36$.

x	y
0	imag.
1	imag.
2	imag.
3	0
4	± 1.7
5	± 2.6
6	± 3.4
etc.	

For negative values of x, the values of y are the same as for the corresponding positive values of x. Hence, the graph is a curve of two branches, ABC and $A'B'C'$, of the species known as the *hyperbola*.



EXERCISE 130

Graph the following:

1. $y = x^2 - 1$.
2. $y = x^2 - 2x - 3$.
3. $y = x^2 - 4x + 4$.
4. $y = x^2 + 3x - 4$.
5. $y = \frac{1}{4}x^2$.
6. $y = x^2 + 1$.
7. $x^2 + y^2 = 16$.
8. $x^2 + y^2 = 9$.
9. $y^2 = 4x - x^2$.
10. $9y^2 + 4x^2 = 36$.
11. $x^2 + 16y^2 = 16$.
12. $16x^2 + y^2 = 16$.
13. $y^2 = 4x$.
14. $y^2 - x^2 = 9$.
15. $x^2 - y^2 = 9$.
16. $xy = 4$.
17. $xy = 1$.
18. $xy = -2$.
19. $x + xy = 1$.
20. $x^2 + (y - 4)^2 = 5$.
21. $9y^2 - x^2 = -9$.
22. $y^2 = 4x + 4$.
23. $x^2 - xy + y^2 = 25$.
24. $y^2 = 4$.
25. $x^2 - 4x + 3 = 0$.

SUG. Show that whatever the value of y, x always $= 1$ or 3; hence the graph is two straight lines parallel with the y-axis.

26. Make up and work an example similar to Ex. 1. To Ex. 7. Ex. 8.

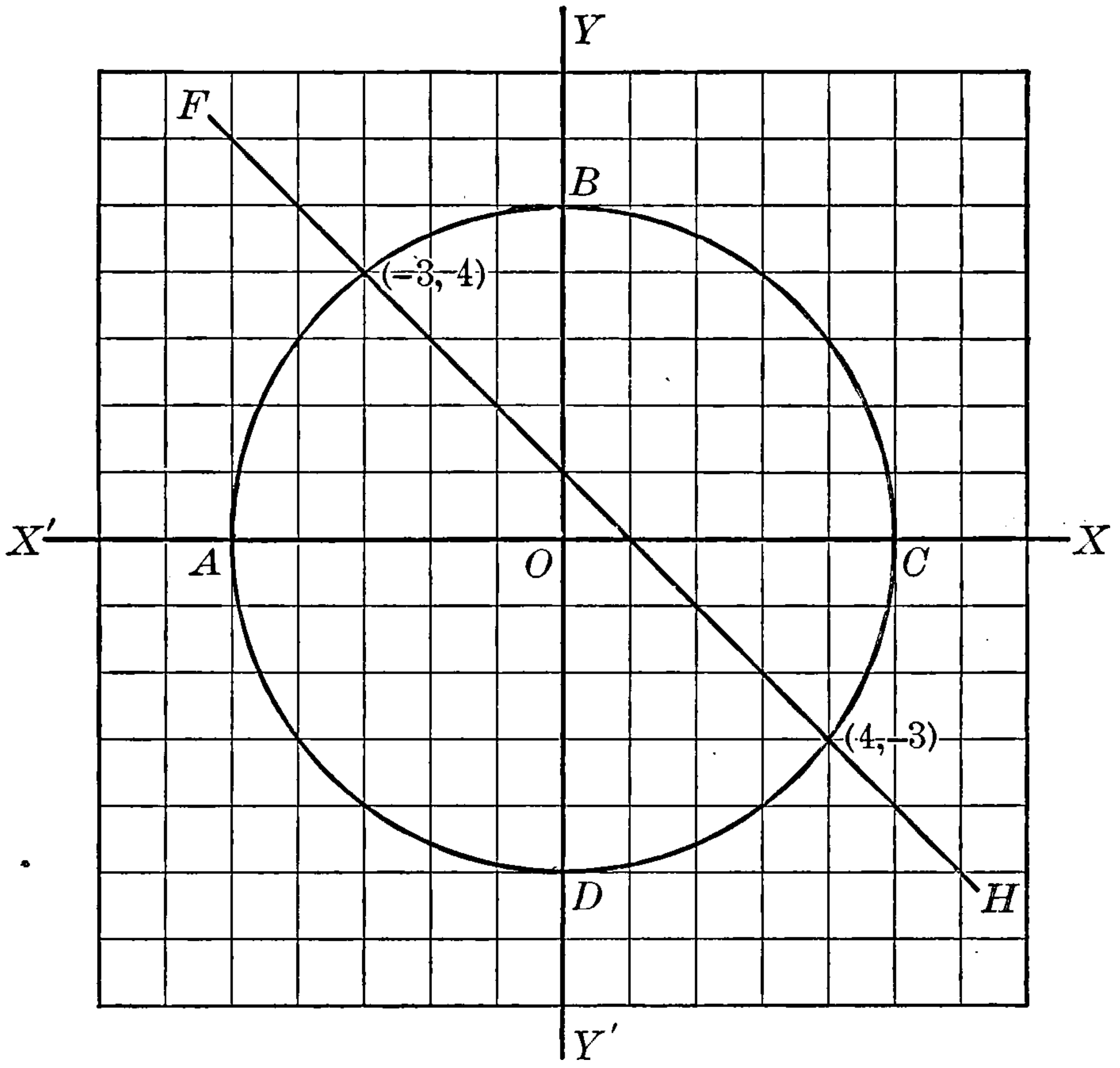
248. Graphic Solution of Simultaneous Quadratic Equations.

Ex. Solve graphically $\begin{cases} x^2 + y^2 = 25. \\ x + y = 1. \end{cases}$

Constructing the graph of $x^2 + y^2 = 25$, we obtain the circle ABC (p. 390). Constructing the graph of $x + y = 1$, we obtain the straight line FH.

Measuring the co-ordinates of the points of intersection of the two graphs, we find the points to be $(4, -3)$ and $-3, 4)$.

These results may be verified by solving the two given simultaneous equations algebraically.



249. Special Cases; Imaginary Roots. Construct the graphs of $x^2 + y^2 = 4$ and $x + y = 3$. You will find that these two graphs do not intersect. Then solve the given equations in the ordinary algebraic way. You will find that the roots are imaginary. If you treat the equations $x^2 + y^2 = 1$ and $4x^2 + 9y^2 = 36$ in the same way, you will obtain a similar result.

In general, *imaginary roots of simultaneous equations correspond to points of non-intersection of the graphs of the given equations.*

Remember that in solving a pair of simultaneous equations, the number of values of x (and also of y) is equal to the sum of the degrees of the two equations. Hence, if two simultaneous equations are both of the second degree, their graphs should intersect in four points; and if their graphs are found to intersect in only two points for instance, the other two points must correspond to imaginary roots.

The pupil may illustrate this by graphing and also solving algebraically $y^2 = 4x$ and $x^2 + y^2 = 25$.

EXERCISE 131

Solve both graphically and algebraically:

1. $y^2 = 4x.$
$y - x = 0.$

2. $y^2 = 4x.$
$y = 2x.$

3. $y^2 = 4x.$
$x + y = 0.$

4. $y^2 = x.$
$x + y = 2.$

5. $y^2 = 2x.$
$y = x - 4.$

6. $x^2 + y^2 = 25.$
$2y - x = 5.$

7. $x^2 + y^2 = 25.$
$y + 2x - 5 = 0.$

8. $x^2 + y^2 = 25.$
$y = x.$

9. $x^2 + y^2 = 25.$
$3y - 4x = 0.$

10. $xy = 3.$
$x + y = 4.$

11. $xy = -3.$
$x + y = 2.$

12. $xy = -2.$
$x + y = -1.$

13. $xy = -2.$
$x + y = 1.$

14. $x^2 + y^2 = 25.$
$xy = 12.$

15. $x^2 + y^2 = 25.$
$x - y = 1.$

16. $x + y = -2.$
$xy = -3.$

17. $y^2 - 4x = 0.$
$3y^2 + 2x^2 = 14.$

18. $x^2 + y^2 - 10x = 0.$
$y = 2x.$

19. $x^2 + y^2 = 16.$
$x^2 + 4y^2 = 43.$

20. $3y^2 - 2x^2 = 12.$
$x^2 + y^2 = 16.$

21. $3x^2 + y^2 = 3.$
$y = x + 2.$

22. $y - x - 6 = 0.$
$x^2 = 6y - y^2.$

23. $4x^2 - 9y^2 = 36.$
$x^2 + y^2 = 1.$

24. $x^2 + y^2 + x + 3y = 18.$
$xy - y = 12.$

25. Solve graphically $x^2 + y = 7.$
$x + y^2 = 11.$

Also try to solve this pair of equations algebraically.

26. Make up and work an example similar to Ex. 1. To Ex. 6. Ex. 10.

250. Graphic Solution of a Quadratic or Higher Equation of One Unknown Quantity. By substituting for y in the first equation, the pair of equations $y = x^2 - 3x + 2$ and $y = 0$ reduces to $x^2 - 3x + 2 = 0$. Accordingly, the graphic solution of an equation like $x^2 - 3x + 2 = 0$ is obtained by solving graphically $y = x^2 - 3x + 2$ and $y = 0$.

In other words, *the roots of a quadratic equation of one unknown quantity,* $ax^2 + bx + c = 0$, *are represented graphically by the abscissas of the points where the graph of* $y = ax^2 + bx + c$ *meets the x-axis.*

Ex. Solve graphically $x^2 - 3x + 2 = 0$.

The graph of $y = x^2 - 3x + 2$ is the curved line ABC of the figure in Art. 247 (p. 387).

This curve crosses the x-axis at the points (1, 0) and (2, 0).

$\therefore x = 1, 2$ *Roots*

The same results are obtained by solving the equation $x^2 - 3x + 2 = 0$ algebraically.

This method of solution applies also to a cubic equation or to an equation of one unknown quantity of any degree.

Thus, to solve the equation $x^3 - 3x^2 + 5x - 2 = 0$, graph the equation $y = x^3 - 3x^2 + 5x - 2$. The abscissas of the points where this graph crosses the x-axis have the same value as the roots of the given equation $x^3 - 3x^2 + 5x - 2 = 0$.

EXERCISE 132

Solve both graphically and algebraically:

1. $x^2 - 4 = 0.$

2. $x^2 - 3x - 4 = 0.$

3. $4x^2 + 8x - 5 = 0.$

4. $x^2 - 6x + 9 = 0.$

5. $x^2 - 4x + 1 = 0$. 6. $x^3 - 2x = 0$.

7. $x^3 - 2x - 1 = 0$.

SUG. Make the algebraic solution by the factorial method.

8. $x^3 - x^2 - 6x = 0$. 9. $x^3 - 2x^2 - 2x + 4 = 0$.

10. $x^4 - 5x^2 + 4 = 0$.

11. Make up and work an example similar to Ex. 1. To Ex. 2. Ex. 10.

SOME APPLIED GRAPHS

251. Wider Application of Graphs. Besides their use in ordinary algebra, graphs may be used to represent the properties of a great variety of functions, in particular those occurring in the various departments of science and in business life.

Sometimes it is found convenient to use a different scale in laying off magnitudes on one axis from that used on the other axis.

EXERCISE 133

1. Graph $C = \frac{5}{9}(F - 32)$, making the scale on the C-axis one half as large as that on the F-axis.

2. A thermometer reads as follows at different hours during the day:

Hour	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 A.M.	1 P.M.	2 P.M.
Temperature .	50°	51°	54°	59°	65°	71°	75°	78°

Hour	3 P.M.	4 P.M.	5 P.M.	6 P.M.	7 P.M.	8 P.M.	9 P.M.	10 P.M.
Temperature .	78°	77°	71°	65°	60°	57°	55°	51°

Construct a graph showing the relation between the temperature above 60° (taken as plus) and that below (taken as minus), and the hour of the day. Then point out some facts to be learned from this graph.

3. Graph $V = \frac{1}{P}$, and on the graph obtained measure the value of V when $P = 1.5$.

4. Construct the graph of $s = 16.1t^2$ making the s-scale but one tenth as large as the t-scale.

5. The average temperature on the first day in each month for the last thirty years in New York City has been as follows. Graph these data.

NEW YORK

Date	Jan. 1	Feb. 1	March 1	April 1	May 1	June 1
Temperature .	31°	31°	35°	42°	54°	64°

Date	July 1	Aug. 1	Sept. 1	Oct. 1	Nov. 1	Dec. 1
Temperature .	71°	73°	69°	61°	49°	39°

The corresponding temperatures in London were as follows:

LONDON

Date	Jan. 1	Feb. 1	March 1	April 1	May 1	June 1
Temperature .	37°	38°	40°	45°	50°	57°

Date	July 1	Aug. 1	Sept. 1	Oct. 1	Nov. 1	Dec. 1
Temperature .	62°	62°	59°	54°	46°	41°

Graph these results on the same paper with the graph of the New York temperatures and then compare the two curves of annual temperature, and give three facts which may be inferred from these curves.

6. The following table shows the number of years which a person having attained a certain age may expect to live. Construct a graph of life expectancy from the data.

Age in Years	0	2	4	6	8	10	20	30
Life Expectancy in Years	38.7	47.6	50.8	51.2	50.2	48.8	41.5	34.3

Age in Years	40	50	60	70	80	90	100
Life Expectancy in Years . .	27.6	21.1	14.3	9.2	5.2	3.2	2.3

From this graph determine your life expectancy at the present time, and also that of several acquaintances of various ages.

7. Graph the growth of the population of the United States using the following table:

Year	1790	1800	1810	1820	1830	1840	1850
Millions	4	5	7	10	13	17	23

Year	1860	1870	1880	1890	1900	1910
Millions.	31	39	50	63	76	92

From your graph determine as accurately as you can the population in 1815. In 1835. In 1895. In 1905.

From your graph determine as nearly as you can in what year the population was 35 milions. 70 millions. 80 millions.

8. The following table gives the amount of $1 at simple interest, and also at compound interest at 4% for 5, 10, 15, 20, etc. years. On the same diagram draw (1) a graph of the amounts at simple interest (2) a graph of the amounts at compound interest.

Years . . .	0	5	10	15	20	25	30	35
Amounts at Simple Int.	$1	$1.20	$1.40	$1.60	$1.80	$2.00	$2.20	$2.40
Amounts at Com. Int.	1	1.22	1.48	1.80	2.19	2.67	3.24	3.95

The amounts of \$1 at 5% for the same periods of time at compound interest are \$1, \$1.28, \$1.63, \$2.08, \$2.65, \$3.39, \$4.32, \$5.52. On the same diagram make a graph of these amounts.

9. The following table gives the pressure for various velocities of the wind:

Velocity of wind in mi. per hr.	10	20	30	40	50	60	70	80	90	100
Pressure in lb. per sq. ft.	1.5	2	4.5	8	12.5	18	24.5	32	40.5	50

Graph the above table of facts. From this graph determine as exactly as you can the pressure when the velocity of the wind is 25 mi. per hour. 45 mi. 65 mi.

10. From the same table determine approximately the velocity of the wind when the wind pressure is 5 lb. per square foot. 10 lb. 30 lb.

11. Graph $y = x^{\frac{3}{2}}$.

12. Graph $y = x^{\frac{2}{3}}$.

13. Construct the parallelogram whose sides are the graphs of the equations $3y - 4x - 13 = 0$, $3y - 4x + 19 = 0$, $y = 3$, $y = -1$. Find the co-ordinates of the vertices of this parallelogram, and also its area.

14. Who first invented graphs, and when?

15. Graphs, or geometric pictures of numerical data, take many different forms beside the linear graphs treated in this book. For instance, the density with which the national banks are distributed over different parts of the country may be indicated by dots scattered over a map. Collect examples of as many different kinds of graphs as possible.

CHAPTER XXII

GENERAL PROPERTIES OF QUADRATIC EQUATIONS

252. Character of the Roots Inferred from the Coefficients. It is important to be able to infer at once from the nature of the coefficients of an equation whether the roots of the equation are equal or unequal, real or imaginary, positive or negative.

Any quadratic equation may be reduced to the form $ax^2 + bx + c = 0$, in which a is positive.

Solving $ax^2 + bx + c = 0$, and denoting the roots by r_1, r_2 (read, r sub-one, r sub-two), we obtain

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

From these expressions we infer that

1. *If* $b^2 - 4ac$ *is* **positive,** *the roots are* real *and* **unequal.**
2. *If* $b^2 - 4ac$ *equals* **zero,** *the roots are* **real** *and* **equal.**
3. *If* $b^2 - 4ac$ *is* **negative,** *the roots are* **imaginary.**

The roots are rational if $b^2 - 4ac$ is a perfect square or zero.

Since the character of the roots is thus determined by the value of $b^2 - 4ac$, this expression is termed the *discriminant* of $ax^2 + bx + c = 0$.

Ex. 1. Determine the character of the roots of the equation, $x^2 - 2x - 1 = 0$.

We have, $a = 1, \quad b = -2, \quad c = -1$

$$b^2 - 4ac = 4 + 4 = 8$$

Hence, the roots are *real* and *unequal.*

Ex. 2. Of $x^2 - 2x + 1 = 0$.

Here, $$a = 1, \quad b = -2, \quad c = 1,$$
$$b^2 - 4ac = 4 - 4 = 0.$$

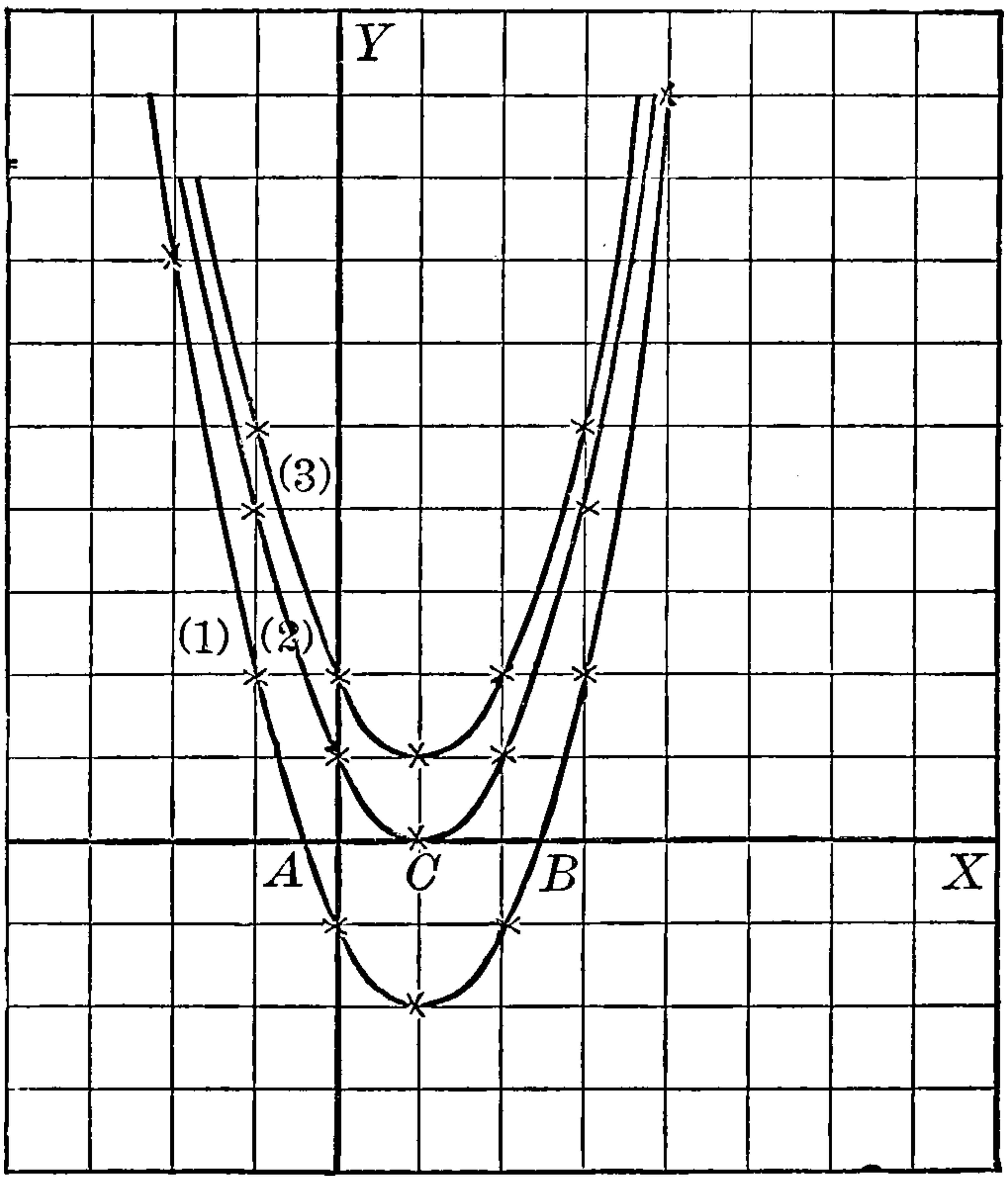
Hence, the roots are *real* and *equal.*

Ex. 3. Of $x^2 - 2x + 2 = 0$.

Here, $$a = 1, \quad b = -2, \quad c = 2,$$
$$b^2 - 4ac = 4 - 8 = -4.$$

Hence, the roots are *imaginary.*

The results obtained in Exs. 1, 2, and 3 may be conveniently illustrated by means of graphs.



It is found that the graph of $y = x^2 - 2x - 1$ is the curve (1) and crosses the x-axis at the two points A and B (corresponding to the two roots of $x^2 - 2x - 1 = 0$).

The graph of $y = x^2 - 2x + 1$ is the curve (2) and meets the x-axis at only one point (corresponding to the two equal roots of $x^2 - 2x + 1 = 0$).

The graph of $y = x^2 - 2x + 2$ is the curve (3) which does not meet the x-axis at all (which illustrates the fact that the roots of the equation $x^2 - 2x + 2 = 0$ are imaginary).

253. Determining Coefficients so that the Roots shall satisfy a Given Condition. It is often possible so to determine the coefficients of an equation that the roots shall satisfy a given condition.

Ex. Find the value of m which will give equal roots for the equation $(m - 1)x^2 + mx + 2m - 3 = 0$.

By Art. 252, 2, in order that the roots may be equal, $b^2-4ac=0$.
In the given equation, $a=m-1,\quad b=m,\quad c=2m-3$.

$$\therefore m^2-4(m-1)(2m-3)=0$$
$$m^2-8m^2+20m-12=0$$
$$7m^2-20m=-12$$
$$m=2,\ \tfrac{6}{7}\ \textit{Ans.}$$

Check. Substituting these values for m in the original equation,

$$x^2+2x+1=0,\qquad x^2-6x+9=0,$$

in each of which equations the roots are equal.

EXERCISE 134

Without solving, determine the character of the roots in

1. $x^2-5x+6=0$.

2. $3x^2-7x-2=0$.

3. $4x^2=4x-1$.

4. $3x^2+2x+1=0$

5. $2x^2-5x+3=0$.

6. $\dfrac{x-1}{3}=x^2-2$.

7. $x=-\dfrac{9x^2+4}{12}$.

8. $6x^2+\frac{25}{6}=10x$.

9. $x=\frac{2}{5}(x^2+1)$.

10. $35x+18+12x^2=0$.

11. $\frac{1}{3}x^2=2x-3$.

12. $7x^2+1=5x$.

13. Determine by inspection the nature of the roots in

$$(1)\quad x^2-4x+2=0$$
$$(2)\quad x^2-4x+4=0$$
$$(3)\quad x^2-4x+6=0$$

Verify your results by use of graphs.

14. Make up and work an example similar to Ex. 13.

15. In $3x^2-2x+1=0$, determine the character of the roots by solving the equation. Now determine their character by the method of Art. 252. Compare the amount of work in the two processes.

Determine the value of m for which the roots of each equation will be equal:

16. $2x^2 - 2x + m = 0.$

17. $mx^2 - 5x + 2 = 0.$

18. $2x^2 - mx + 12\frac{1}{2} = 0.$

19. $4x^2 + \frac{1}{9} = mx.$

20. $(m + 1)x^2 + mx = 1.$

21. $(m + 1)x^2 + 3m = 12x.$

22. $(m + 1)x^2 + (m - 1)x + m + 1 = 0.$

23. $\dfrac{2x^2 + 3x + 2}{7 + 3x - x^2} = \dfrac{1}{m}.$

24. What is meant by a root of an equation?

25. In Ex. 16 for what values of m are the roots imaginary? Real and unequal?

26. Answer the same questions for Ex. 17. Ex. 20.

27. Show that if one root of a quadratic equation is imaginary the other root must be imaginary also.

28. State and prove a similar fact concerning irrational roots.

29. For what values of p does the equation $x^2 + (7p - 3)x - (5p + 10) = 0$ have a zero root? Find the other root.

SUG. If one root is zero, what does the product of the roots equal?

30. How many of the above examples can you work at sight?

254. Relation between Roots and Coefficients. In Art. 252 a method was obtained of inferring from the coefficients of a quadratic equation, the *qualitative* nature of the roots. A more exact, or *quantitative*, relation between the roots and coefficients will now be obtained. This relation will enable us in any given equation to determine the sum or product (and often other functions) of the roots, without the labor of solving the equation.

Dividing both members of $ax^2 + bx + c = 0$ by a, we obtain an equation in the form $x + px + q = 0$.

Solving this equation and denoting its roots by α and β,

$$\alpha = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } \beta = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

Adding the roots, $\alpha + \beta = \frac{-2p}{2} = -p$

Multiplying the roots, $\alpha\beta = \frac{p^2 - (p^2 - 4q)}{4} = q$

Hence, in general,

(1) *The sum of the roots of* $x^2 + px + q = 0$ *equals* $-p$, *or the coefficient of* x *with the sign changed;*

(2) *The product of the roots equals the known term* q.

Ex. 1. Without solving the equation, find the sum and also the product of the roots of $5(1 - 2x) = 3x^2$.

The given equation reduces to $x^2 + \frac{10}{3}x - \frac{5}{3} = 0$.

Hence, sum of roots $= -\frac{10}{3}$, product of roots $= -\frac{5}{3}$ *Ans.*

Ex. 2. Form the equation whose roots are $\frac{-1 \pm \sqrt{-3}}{2}$.

The roots are $\frac{-1 + \sqrt{-3}}{2}, \frac{-1 - \sqrt{-3}}{2}$

Hence, sum of roots $= \frac{-1 + \sqrt{-3} - 1 - \sqrt{-3}}{2} = \frac{-2}{2} = -1$

Product of roots $= \frac{1 - (-3)}{4} = \frac{1 + 3}{4} = 1$

Hence, $x^2 + x + 1 = 0$ is the required equation.

Checks for the above example may be obtained by solving the equations obtained.

255. Factoring a Quadratic Expression. Any quadratic expression may be factored by letting the given expression equal zero, and using the property stated in Art. 254.

Ex. Factor $3x^2 - 4x + 5$.

Take $$3(x^2 - \tfrac{4}{3}x + \tfrac{5}{3}) = 0$$

Solve $$x^2 - \tfrac{4}{3}x + \tfrac{5}{3} = 0$$

Whence, $$x = \frac{2 \pm \sqrt{-11}}{3}.$$

Hence, the factors of $3x^2 - 4x + 5$ are

$$3\left(x - \frac{2 + \sqrt{-11}}{3}\right)\left(x - \frac{2 - \sqrt{-11}}{3}\right) \textit{ Ans.}$$

EXERCISE 135

Find, by inspection, the sum and product of the roots in each of the following equations:

1. $x^2 + 3x + 5 = 0$.
2. $x^2 - x + 7 = 0$.
3. $x^2 - 5x = 10$.
4. $2x^2 - 6x - 3 = 0$.
5. $x + 1 = \dfrac{x - 5x^2}{x - 1}$.
6. $a^2x^2 - ax + 2 = 0$.
7. $5x - 4x^2 = 1$.
8. $3 - 7x = 11x^2$.
9. $\dfrac{x}{a} = \dfrac{x - a}{4x + 1}$.
10. $1 - 2cx - 2ax^2 = 3c$.

Form the equations whose roots are

11. 2, 3.
12. 3, -2.
13. -1, -5.
14. 5, .04.
15. -1, $-\frac{2}{3}$.
16. $\frac{3}{2}$, $-\frac{2}{3}$.
17. .08, $-.2$.
18. ab, $-a$.
19. $\dfrac{a}{b}$, $-\dfrac{b}{a}$.
20. $1 + \sqrt{2}$, $1 - \sqrt{2}$.
21. $-3 \pm \sqrt{3}$.
22. $\dfrac{2 \pm \sqrt{2}}{2}$.
23. $\dfrac{1 \pm \sqrt{-1}}{2}$.
24. $\dfrac{-2 \pm \sqrt{-2}}{2}$.
25. $\frac{1}{2}a \pm c\sqrt{-b}$.

26. Form an equation whose roots are $2 \pm i$ (see p. 335).

27. Form an equation whose roots are 3, i, $-i$.

28. If one root of the equation $x - \frac{13}{5}x - \frac{6}{5} = 0$ is 3, find the other root in two different ways.

29. If one root of the equation $3x^2 - 4x + 2 = 0$ is $\frac{2 + \sqrt{-2}}{3}$, find the other root in three different ways.

30. Form the equation in which one root is $-\frac{2}{3}$ and the product of the roots is $-\frac{8}{3}$.

31. Make up and work an example similar to Ex. 28. Also to Ex. 30.

Factor:

32. $3x^2 - 10x - 8$.

33. $x^2 + 2x - 1$.

34. $x^2 - x - 1$.

35. $x^2 + 14 - 6x$.

36. $25x^2 + 2 - 30x$.

37. $3x - 3x^2 - 1$.

38. If r and s represent the roots of $3x^2 - 8x + 5 = 0$, find, without determining the actual roots, the values of $r + s$; rs; $r^2 + s^2$; $r - s$; $r^2 - s^2$; $r^3 + s^3$; $\frac{1}{r} + \frac{1}{s}$; $\frac{1}{r} - \frac{1}{s}$; $\frac{1}{r^2} - \frac{1}{s^2}$.

39. Find the values of the same expressions for the equation $2x^2 - 9x + 7 = 0$. Also for $6x^2 - x - 12 = 0$.

40. Find the values of the same expressions for the equation $ax^2 + bx + c = 0$. Also for the equation $x^2 + px + q = 0$.

41. If m and n represent the roots of the equation $10x^2 + 9x - 7 = 0$, form that equation whose roots shall be mn and $m + n$. $m - n$ and $\frac{1}{m} + \frac{1}{n}$.

42. In $ax^2 + bx + c = 0$, if $c = a$ show that one root is the reciprocal of the other.

43. Make up and work an example similar to Ex. 7. To Ex. 23. Ex. 32. Ex. 39.

44. How many of the examples in this Exercise can you work at sight?

45. Practice oral work with exponents as in Exercise 93 (p. 303).

CHAPTER XXIII

RATIO AND PROPORTION

Ratio

256. The Ratio of two algebraic quantities is their exact relation of magnitude. It is the indicated quotient of the one quantity divided by the other, expressed either in the form of a fraction or by the symbol : placed between the two quantities.

Thus, the ratio of a to b is expressed as $\frac{a}{b}$, or as $a : b$.

257. Utility of Ratios. Ratios have the same uses as fractions (see Art. 110, p. 166). These uses are extended by selecting important kinds of ratios, naming them (see Art. 259), and working out their properties once for all. Also properties of equal ratios are worked out once for all and stated in such a form as facilitates their application to problems.

258. The Terms of a Ratio are the two quantities compared. The *antecedent* is the first term. The *consequent* is the second term.

The terms of a ratio must be expressed in terms of a common unit. Thus, to express the ration of 3 qt. to 2 bu. either the quarts must be expressed as bushels or the bushels as quarts.

If two quantities, as 5 in. and 2 bu., cannot be expressed in terms of the same unit, no ratio between them is possible.

259. Kinds of Ratio. An **inverse** ratio is a ratio obtained by interchanging antecedent and consequent.

Thus, the direct ratio of a to b is $a : b$; the inverse ratio of the same quantities is $b : a$.

A **compound** ratio is one formed by taking the product of the corresponding terms of two given ratios.

Thus, $ac : bd$ is the ratio compounded of $a : b$ and $c : d$.

A **duplicate** ratio is formed by compounding a ratio with itself.

Thus, the duplicate ratio of $a : b$ is $a^2 : b^2$. In like manner, the **triplicate** ratio of $a : b$ is $a^3 : b^3$.

A **commensurable ratio** is a ratio that can be expressed in terms of two integers.

Thus, $\frac{2\frac{1}{2}}{5\frac{1}{4}}$ equals $\frac{5}{2} \div \frac{21}{4}$, or $\frac{10}{21}$. Or $\frac{2\frac{1}{2}}{5\frac{1}{4}}$ is a commensurable ratio.

An **incommensurable ratio** is a ratio which cannot be expressed in terms of two integers.

Thus, $\frac{\sqrt{2}}{5}$ equals $\frac{1.4142 +}{5}$. The fraction in the numerator cannot be completed so that the numerator and denominator can be expressed as a ratio of integers in terms of the same unit; hence $\frac{\sqrt{2}}{5}$ is an incommensurable ratio.

The properties of incommensurable ratios are obtained from those of commensurable ratios by an indirect method not discussed in this book.

260. Fundamental Property of Ratios. *If both antecedent and consequent of a ratio are multiplied or divided by the same quantity, the value of the ratio is not changed.*

For, since
$$\frac{a}{b} = \frac{ma}{mb}$$
$a : b$ has the same value as $ma : mb$.

EXERCISE 136

Simplify each of the following ratios:

1. 1 ft. 8 in. : 4 in.

2. $7\frac{1}{2} : \frac{5}{3}$.

3. 1 gal. : 1 qt. 1pt.

4. $37\frac{1}{2}\% : 12\frac{1}{2}\%$.

5. $x^2 - 1 : (x+1)^2$.

6. $\frac{a^2 - b^2}{(a+b)^2} : \frac{(b-a)^2}{a^2+b^2}$.

7. $.01x^2 : .5x^3$.

8. $(a^5)^{10} : a^5 \cdot a^{10}$.

9. Find the ratio of $4x^2$ to $(4x)^2$, when $x = 6$. When $x = \frac{1}{2}$.

10. In the year 1910 an automobile went 1 mi. in $27\frac{1}{3}$ sec. How many feet per second was this?

11. Find the ratio of the area of a rectangle 3 yd. long and 2 ft. wide to the area of a rectangle $2\frac{1}{2}$ ft. long and 18 in. wide.

Find the ratio of x to y from

12. $7x - 3y = 4x + y$.

13. $4x - 5y : 5x - 4y = \frac{2}{3}$.

14. $\frac{3x - 2y}{4x - 3y} = \frac{a}{b}$.

15. $x^2 + 6y^2 = 5xy$.

16. A horse can pull 2 tons on a level macadam road, 15 tons on a level iron track, and 70 tons on a canal. Find the ratio of each pair of these numbers.

17. The death rate in London in the year 1700 was approximately 80 per thousand. In the year 1908 it was 14.3 per thousand. In a population of 5,000,000 people how many lives are saved per annum by the diminished death rate?

18. What is meant by the specific gravity of iron (or of any material)? If a cubic foot of water weighs 62.5 lb., find the weight of brick whose specific gravity is 2.3 and which fill a wagon $6' \times 3' \times 2'$.

19. If the weight of the human brain is $\frac{1}{40}$ the weight of the body, while the blood in the brain is $\frac{1}{6}$ of all the blood in the body, the density of the blood in the brain is how many

times that in the body as a whole? Of that in the rest of the body?

20. If a ratio is less than unity, does adding the same quantity to both the terms of the ratio increase or diminish the value of the ratio? By how much if the ratio is $\frac{a}{b}$ and c be added to both terms?

21. Answer the same questions if the given ratio is greater than unity.

22. Find out, if you can, what is meant by a nutritive ratio and make up and work three examples concerning such ratios.

23. Make up and work an example similar to Ex. 3. To Ex. 10. Ex. 11.

24. How many of the examples in this Exercise can you work at sight?

PROPORTION

261. A Proportion is an expression of the equality of two or more equal ratios; as $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d$.

The above proportion is read "a is to b as c is to d."

262. Terms of a Proportion. The four quantities used in a proportion are called its *terms*, or *proportionals*.

The first and third terms are the *antecedents*.

The second and fourth terms are the *consequents*.

The first and last terms are the *extremes*.

The second and third terms are the *means*.

In $a : b = c : d$, d is a *fourth proportional* to a, b, and c.

263. A Continued Proportion is one in which each consequent and the next antecedent are the same; as

$$a : b = b : c = c : d = d : e.$$

In the continued proportion $a : b = b : c$, b is called a *mean proportional* between a and c; c is called a *third proportional* to a and b.

Two proportions of the form $x : y = a : b$, and $y : z = b : c$ may be combined in the form $x : y : z = a : b : c$.

264. Equal Products made into a Proportion. *If the product of two quantities is equal to the product of two other quantities, either two may be made the means, and the other two the extremes of a proportion.*

For, if $$ad = bc$$

Dividing by bd, $$\frac{a}{b} = \frac{c}{d} \qquad \text{(Art. 15, 6)}$$

$$\therefore a : b = c : d$$

265. Fundamental Property of Proportion. For algebraic purposes, the fundamental property of a proportion is that

The product of the means is equal to the product of the extremes.

For, if $$a : b = c : d$$

then $$\frac{a}{b} = \frac{c}{d}$$

Multiplying by bd, $$ad = bc \qquad \text{(Art. 15, 5)}$$

In like manner, if $$a : b = b : c$$

$$b^2 = ac \qquad \therefore b = \sqrt{ac}$$

This property enables us to convert a proportion into an equation, and to solve a given proportion by solving the equation thus obtained.

EXERCISE 137

Find a mean proportional between

1. $3ab^2$ and $12a^3$. **2.** $3\frac{1}{5}$ and $2\frac{2}{9}$. **3.** $(a-x)^2$ and $(a+x)^2$.

4. $\dfrac{3x^2-5x-12}{3x^2+5x}$ and $\dfrac{3x^2+4x}{3x^2-4x-15}$.

5. $\dfrac{2\sqrt{6}+5\sqrt{3}}{3\sqrt{2}-4}$ and $\dfrac{3\sqrt{6}-4\sqrt{3}}{8\sqrt{2}+20}$.

Find a fourth proportional to

6. $2a, 3b, 4ac$.

7. $x^3, xy, 3x^2$.

8. $\frac{2}{3}, \frac{5}{7}, \frac{2}{15}$.

9. $a-1, a, 1$.

Find a third proportional to

10. x and 5.

11. .4 and .08.

12. $(a+1)^2$ and a^2-1.

13. $a-\dfrac{1}{a}$ and $\dfrac{1}{a}-1$.

14. How many answers to each of Exs. 1–5? To each of Exs. 6–9? Exs. 10–13?

Solve and check:

15. $2x+3:3x-1=3x+1:2x+1$.

16. $x+5:3-x=10+3x:x-10$.

17. $3x+5:5x+11=7-x:-3x$.

18. $x^2-4:x^2-x+3=x+2:2x+3$.

19. What numbers must be added to each of the terms of $\frac{2}{5}$ to make the value of the fraction $\frac{2}{3}$?

20. A baseball nine has won 17 games out of 25. How many straight games will it have to win to make the games won equal $\frac{3}{4}$ of the games played?

21. What number added to each of the numbers 3, 7, 15, and 25 will give results which are in proportion?

22. The horse-power generated by a stream falling over a dam is proportional to the height of the dam. If on a certain stream a dam 5 ft. high generates 200 H. P., how much higher must the dam be made in order to get 280 H. P.? 350 H. P.?

23. What short way is there of determining whether a given proportion is correct? Illustrate by examples.

24. Test the following proportions: $1:-3=-3:1$. $-1:1=1:-1$.

25. Convert each of the following into a proportion:

(1) $3\times 4=6\times 2$.
(3) $x^2=a^2-b^2$.
(2) $pq=ab$.
(4) $15=x^2$.

26. Separate $a:b:c=4:5:6$ into two proportions.

27. Combine $a:b=2:3$ and $b:c=3:5$ as a single statement.

28. Make up and work an example similar to Ex. 27.

29. Separate 1200 into two parts which shall be in the ratio of 2 to 3.

30. Separate 1200 into three parts which shall be in the ratio of 3, 4, and 5.

31. Make up and work an example similar to Ex. 30.

32. That a door may look well, its height should be to its width approximately as $7:5$. If a door is to be 6 ft. 9 in. high, how wide should it be?

33. In a certain year the profits of a given business were $39,260. Divide these profits into two parts which shall be as 9 to 4. Also into three parts, as 8, 3, 2.

34. In the steepest part of the Mt. Washington railway (Jacob's Ladder), there is a rise of 13 in. in one yard of track. What would be the rise in a mile of track at the same rate?

35. If the area of Rhode Island is 1250 sq. mi., of New Jersey 7800 sq. mi., and of New York 49,000 sq. mi., by how much does the area of New Jersey differ from a mean proportional between the other two areas?

36. The lengths of the Hudson, Ohio, and Mississippi rivers are respectively 280, 950, and 3160 miles. By how much does the length of the Ohio differ from a mean proportional between the lengths of the other two rivers?

37. If the weights of a man, horse, and elephant are respectively 150 lb., 1000 lb., and $2\frac{1}{2}$ T., how much does the last of these numbers differ from a third proportional to the other two?

38. If the diameter of the moon, the distance of the moon, the diameter of the sun, and the distance of the sun are taken as 2160, 240,000, 860,000, and 93,000,000 mi. respectively, how much does the last number differ from a fourth proportional to the other three numbers in the order given? By what per cent does it differ?

39. In sterling silver, the amount of the silver is .925 of the entire weight of the metal. 500 ounces of pure silver will make how many ounces of sterling silver? (What other metal is added to pure silver to make sterling silver, and why is it added?)

40. If a given piece of ground can be divided up into 60 building lots each 30 ft. wide, how many lots 40 ft. wide would it make?

SUG. If x denote the number of lots 40 ft. wide,

$$30 \times 60 = 40 \times x$$

or,

$$x : 60 = 30 : 40$$

This problem can be solved either from the equation or from the proportion.

A proportion of this kind is termed an *inverse proportion.*

41. If 2400 shingles 4 in. wide are needed in building a house, how many 3 in. shingles would be needed?

42. If 15 yd. of cloth 36 in. wide are used in making a dress, how many yards 48 in. wide would be needed?

43. If a trolley company reduces the hours of its conductors from 12 to 10 hours per day, by what per cent must it increase the number of its conductors?

44. Make up and work an example similar to Ex. 12. To Ex. 22. Ex. 41.

45. Practice oral work with radicals as in Exercise 108.

266. Transformations of a Proportion. Before converting a proportion into an equation, the proportion may often be simplified by the use of one or more of the following principles:

If $a : b = c : d$, then

1. $a : c = b : d$ (called *alternation*).
2. $b : a = c : d$ (*inversion*).
3. $a + b : b = c + d : d$ (*addition*).
4. $a - b : b = c - d : d$ (*subtraction*).
5. $a + b : a - b = c + d : c - d$ (*addition and subtraction*).

For, from $a : b = c : d$, we have $ad = bc$ (Art. 265); whence we obtain 1 and 2 (Art. 264).

Also $\frac{a}{b} = \frac{c}{d}$, whence $\frac{a}{b} + 1 = \frac{c}{d} + 1$ (Art. 15, 3).

Whence, $\frac{a + b}{b} = \frac{c + d}{c}$

Let the pupil prove 4 in like manner, and obtain 5 from 3 and 4.

Ex. 1. Solve $x^2 - 2x + 3 : x^2 + 2x - 3 = 2x^2 - x - 3 : 2x^2 + x + 3$.

By addition and subtraction, $$\frac{2x^2}{-4x + 6} = \frac{4x^2}{-2x - 6}$$

Divide by $\frac{2x^2}{-2}$, $\frac{1}{2x-3}=\frac{2}{x+3}$

$$\therefore\ x+3=4x-6$$

$$x=3\ \textit{Ans.}$$

The factor $2x^2$ also gives the roots $x=0,\ 0$.

CHECK. For $x=3$.

$$\frac{x^2-2x+3}{x^2+2x-3}=\frac{9-6+3}{9+6-3}=\frac{6}{12}=\frac{1}{2}$$

$$\frac{2x^2-x-3}{2x^2+x+3}=\frac{18-3-3}{18+3+3}=\frac{12}{24}=\frac{1}{2}$$

Let the pupil check the work for $x=0$.

Ex. 2. Solve $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}$.

By addition and subtraction, $\frac{2\sqrt{x+1}}{2\sqrt{x-1}}=\frac{4x+1}{4x-3}$

Simplifying and squaring, $\frac{x+1}{x-1}=\frac{16x^2+8x+1}{16x^2-24x+9}$

By addition and subtraction, etc., $\frac{x}{1}=\frac{16x^2-8x+5}{16x-4}$

Hence, $16x^2-4x=16x^2-8x+5$

$$x=\tfrac{5}{4}\ \textit{Ans.}$$

Let the pupil check the work.

267. Given some proportion (or equality of several equal ratios), as $a:b=c:d$, a required proportion is often readily proved by *taking* $\frac{a}{b}=\frac{c}{d}=r$ (hence, $a=br$, $c=dr$), *and substituting for a and c in the required proportion.*

Ex. Given, $a:b=c:d$,

Prove $2a^3+3ab^2:2a^3-3ab^2=2c^3+3cd^2:2c^3-3cd^2$.

Let $\frac{a}{b}=\frac{c}{d}=r$ $\therefore a=br,\ c=dr.$

Substitute in each ratio the values $a=br$, $c=dr$.

$$\text{I.}\quad \frac{2a^3+3ab^2}{2a^3-3ab^2}=\frac{2b^3r^3+3b^3r}{2b^3r^3-3b^3r}=\frac{b^3r(2r^2+3)}{b^3r(2r^3-3)}=\frac{2r^2+3}{2r^3-3};$$

$$\text{II.}\quad \frac{2c^3+3cd^2}{2c^3-3cd^2}=\frac{2d^3r^3+3d^3r}{2d^3r^3-3d^3r}=\frac{d^3r(2r^2+3)}{d^3r(2r^2-3)}=\frac{2r^2+3}{2r^2-3}.$$

$\therefore \dfrac{2a^3+3ab^2}{2a^3-3ab^2}=\dfrac{2c^3+3cd^2}{2c^3-3cd^2}$, since they are each equal to the same expression.

Hence, a given expression may be proved to be identical with another expression either (1) by reducing the first expression directly to the form of the second or (2) by reducing both expressions to a common third form.

268. Composition of Several Equal Ratios. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent.*

Given, $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\dfrac{g}{h}$

Let each of the equal ratios equal r.

Then $\dfrac{a}{b}=r,\ \dfrac{c}{d}=r,\ \dfrac{e}{f}=r,\ \dfrac{g}{h}=r$

$$\therefore a=br,\quad c=dr,\quad e=fr,\quad g=hr$$

Adding the last series of equalities,

$$a+c+e+g=(b+d+f+h)r$$

$$\therefore \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}$$

$$\therefore a+c+e+g : b+d+f+h = a : b$$

EXERCISE 138

Solve:

1. $x^2+2x-1 : x^2+2x+5 = 2x+1 : 2x-5.$
2. $x^3-3x^2+5 : x^3+3x^2-5 = x^2+2 : x^2-2.$
3. $2x^3-8x^2-3x+1 : 2x^3-10x^2+3x-1 = x^2+11 : x^2-11.$

4. $\sqrt{3x+1} : 2\sqrt{2x-1} = \sqrt{x-1} : \sqrt{x+4}$.

5. $\dfrac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \dfrac{\sqrt{a}+1}{\sqrt{a}-1}$.

6. $\dfrac{2x^3-3x^2+x+1}{2x^3-3x^2-x-1} = \dfrac{3x^3-x^2+5x-13}{3x^3-x^2-5x+13}$.

7. $\dfrac{3+\sqrt{2x+3}}{5-\sqrt{2x+3}} = \dfrac{4+\sqrt{x+1}}{4-\sqrt{x+1}}$.

8. $\dfrac{\sqrt{5}+\sqrt{5+x}}{\sqrt{5}-\sqrt{5+x}} = \dfrac{\sqrt{2}+\sqrt{x-2}}{\sqrt{2}-\sqrt{x-2}}$.

9. $\dfrac{3a+\sqrt{4x-3a^2}}{5a-\sqrt{4x-3a^2}} = \dfrac{a+\sqrt{x+a^2}}{3a-\sqrt{x+a^2}}$.

10. $8y-6x : x+y-1 = 5-3x : 4-y = 7 : 4$.

11. $\begin{cases} x+y : y-x = a : 1. \\ xy-3 : x-1 = a+2 : 1. \end{cases}$

If $a : b = c : d$, prove

12. $a^2 : c^2 = ab : dc$.

13. $a^2 : b^2 = a^2+c^2 : b^2+d^2$.

14. $ac : bd = (a+c)^2 : (b+d)^2$.

15. $(a-c)^2 : (b-d)^2 = a^2+c^2 : b^2+d^2$.

16. $a : b = \sqrt{a^2+3c^2} : \sqrt{b^2+3d^2}$.

17. $2a^2+3ab : 3ab-4b^2 = 2c^2+3cd : 3cd-4d^2$.

18. $a^2-ab+b^2 : \dfrac{a^3-b^3}{a} = c^2-cd+d^2 : \dfrac{c^3-d^3}{c}$.

If a, b, c, d are in continued proportion, prove

19. $a : c-d = b^2 : bd-cd$.

20. $a : c = a^2+b^2+c^2 : b^2+c^2+d^2$.

21. $a : d = a^3+2b^3+3c^3 : b^3+2c^3+3d^3$.

22. If $(a+b)(c-d)+(b+c)(d-a)=cd-ab$, prove that $a:b=c:d$.

23. If $(a+b-3c-3d)(2a-2b-c+d)=(2a+2b-c-d)(a-b-3c+3d)$, prove that $a:b=c:d$.

24. Find two numbers in the ratio of 2 to 5, such that when each is increased by 5 they shall be as 3 to 5.

25. Find two numbers, such that if 7 be added to each they will be in the ratio of 2 to 3; and if 2 be subtracted from each, they will be in the ratio of 1 to 3.

26. Separate 32 into two parts, such that the greater diminished by 11 shall be to the less, increased by 5, as 4 to 9.

27. Separate 12 into two parts, such that their product shall be to the sum of their squares as 2 to 5.

28. Is a proportion true after the same number has been added to all of its terms? Give a numerical illustration.

29. If $a:b=c:d$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

30. If $a:b=c:d$, prove that $b:a=\frac{1}{c}:\frac{1}{d}$.

31. A and B are in business and their respective shares of the profits are as 2 to 3. If the profits for a certain year are \$16,000, and during the year A takes out \$1200 and B \$1000, at the end of the year how much of the profits does each receive?

32. During the American Civil War, in the Northern armies 224,000 men died of disease and 110,000 of wounds received in battle. Owing to improved sanitary methods in the Russo-Japanese war, in the Japanese armies 27,000 men died of disease while 59,000 died of wounds. Approximately how many lives were saved in the Japanese armies by the use of improved sanitary methods?

33. An active walker goes 4 mi. an hour. Sensation travels along a nerve at the average rate of 120 ft. per sec. Find the velocity of a rifle bullet which is a third proportional to the velocities just named.

34. The velocity of the earth in its orbit is 18 mi. per sec.; of a message on a submarine cable 2480 mi. per sec.; and of light 186,300 mi. per sec. How much is the middle one of these velocites from a mean proportional between the other two?

35. The sun's distance from the earth is 93,000,000 mi. and light from a star travels 5,900,000,000,000 mi. in a year (called a "light year"), show that the following proportion is approximately correct:

1 inch : 1 mile = sun's distance : 1 light year.

36. If the rear and fore wheels of a wagon are respectively a and b feet in circumference, how many rotations does the rear wheel make while the fore wheel rotates p times?

37. If $\frac{x-y}{1}=\frac{y-z}{m}=\frac{z-x}{n}$, and x, y, and z are unequal, show that $l+m+n=0$.

38. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that $a:b=c:d$.

39. If $a:b=c:d=e:f$, prove that

$$c:d=\sqrt{a^2+4ac+5c^2}:\sqrt{b^2+4bd+5d^2}.$$

40. If a, b, c, and d are in continued proportion, prove that

$$\frac{a^2}{b^2}=\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{a^2-c^2}{b^2-d^2}.$$

41. Solve Exs. 26–30 (pp. 241–242) by aid of the graphical method given in Ex. 2, pp. 260–261, and by the principles of proportion.

42. Make up and work an example similar to Ex. 2. To Ex. 13. Ex. 31.

43. Practice oral work with imaginaries as in Exercise 110 (p. 339).

MEANING OF THE RATIO FORMS $\frac{0}{a}$, $\frac{a}{0}$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

269. The Meaning of $\frac{0}{a}$ has been made clear in Exs. 30–32, p. 79. The same result has also been made evident in the process of constructing certain graphs.

In general,

Zero divided by any number (except zero) gives zero.

270. Infinity is a number (or quantity) greater than any assignable (or definitely expressible) number.

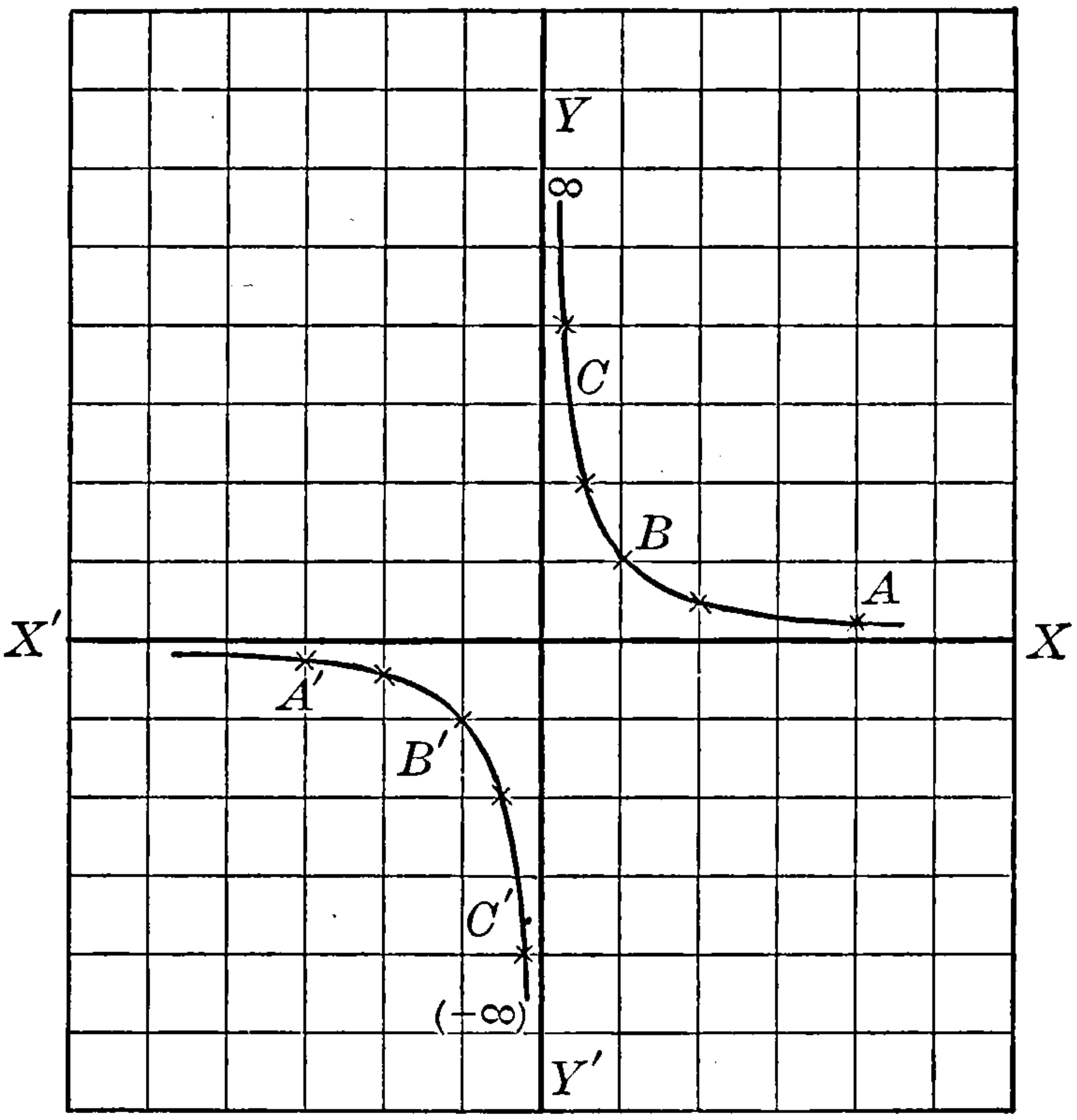
The symbol for infinity is ∞.

The symbol $\doteq$ means "approaches."

271. The Meaning of $\frac{a}{0}$ is made clear most readily by the use of graphs.

Ex. Graph $y = \frac{1}{x}$.

x	y
4	$\frac{1}{4}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2
$\frac{1}{4}$	4
$\frac{1}{8}$	8
etc.	
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4
$-\frac{1}{8}$	-8
etc.	



Graphing $y = \frac{1}{x}$ for the positive values of x, we obtain the branch ABC (p. 419).

From the diagram it is evident that as the value of x becomes smaller, that of $y\left(\text{or } \frac{1}{x}\right)$ increases, and as $x \doteq 0$, $y\left(\text{or } \frac{1}{x}\right) \doteq \infty$.

If $y = \frac{1}{x}$ is graphed for the negative values of x, in like manner as $x = 0$, $y\left(\text{or } \frac{1}{x}\right) \doteq -\infty$.

Hence, in general,

As the value of the denominator of any fraction approaches zero (the value of the numerator being finite), the value of the fraction approaches infinity (or negative infinity).

This statement is often abbreviated into the form $\frac{a}{0} = \infty$.

272. The Meaning of $\frac{0}{0}$ is best shown algebraically.

Ex. Find the value of $\frac{x^2-1}{x-1}$ when $x = 1$.

If we substitute the given value of x directly in the given fraction, we obtain

$$\frac{x^2-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

If, however, we simplify the fraction before substituting and then substitute 1 for x, we obtain

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 = 1+1 = 2$$

Hence, in this case $\frac{0}{0}$ represents the value 2.

Show in a similar manner, that when $x = 1$, the value of $\frac{x^3-1}{x-1}$ is 3. Also that the value of $\frac{x^4-1}{x-1}$ is 4.

Hence, the value of $\frac{0}{0}$ varies with circumstances, or as it is usually expressed

$$\frac{0}{0} \textit{ is indeterminate.}$$

273. The Meaning of $\frac{\infty}{\infty}$ is best shown algebraically.

Ex. Find the value of $\dfrac{1}{x-1} \div \dfrac{1}{x^2-1}$, when $x = 1$.

By direct substitution, $\dfrac{1}{x-1} \div \dfrac{1}{x^2-1} = \dfrac{1}{0} \div \dfrac{1}{0} = \dfrac{\infty}{\infty}$ (see Art. 271).

But if the given expression is simplified before the substitution for x is made,

we have $\dfrac{1}{x-1} \div \dfrac{1}{x^2-1} = \dfrac{1}{x-1} \times \dfrac{x^2-1}{1} = x + 1 = 1 + 1 = 2.$

Hence, in this case $\frac{\infty}{\infty}$ stands for the value 2.

Show, in like manner, that ∞ may stand for 3, 4, or any number. We express the result arrived at as follows:

$\frac{\infty}{\infty}$ *is indeterminate.*

The above results for the meaning of $\dfrac{0}{a}$, $\dfrac{a}{0}$, $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$ might be obtained by purely logical methods, but the thorough discussion of these methods lies beyond the scope of this book.

EXERCISE 139

1. Graph $y = \dfrac{2}{x}$ and thus find the value of $\dfrac{2}{0}$ and $\dfrac{2}{\infty}$.

2. Graph $y = \dfrac{x}{x-1}$. In this process what special ratio is evaluated?

3. Graph $y = \dfrac{x}{2-x}$ and make a similar inference.

By an algebraic process find the value of

4. $\dfrac{x^2-4}{x-2}$ when $x = 2$.

5. $\dfrac{x^3-8}{x-2}$ when $x = 2$.

6. $\dfrac{a^5-1}{a-1}$ when $a = 1$.

7. $\dfrac{a-2}{a^2-a-2}$ when $a = 2$.

Find the value of

8. $\dfrac{x^5}{2}$ when $x = 0$.

9. $\dfrac{1}{x^5}$ when $x = 0$.

10. x^{-5} when $x = 0$. **11.** $7^{\frac{1}{x}}$ when $x = 0$.

12. $\dfrac{x^2}{x-1}$ when $x = 0$. When $x = 1$.

13. $5 \times 0 - \frac{0}{3}$. **14.** $3 \times 0 - \frac{0}{5} + \frac{5}{0}$.

15. $5x - \dfrac{3}{x} + x(x-1)$ when $x = 1$. Also when $x=0$.

16. $a \times 0 + b \times 0 - c \times 0$.

17. $\sqrt[x]{5}$ when $x = 0$.

18 $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a-x}}$ when $x = a$.

19. What is the limit of $\dfrac{3 + 7^{-n}}{2 + \frac{1}{n}}$ when n is increased indefinitely?

20. Find the value which $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ approaches when $b \doteq 0$. When $c \doteq 0$. $a \doteq 0$.

21. Find the same values for $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

22. Who invented ∞ as the sign for infinity?

23. Make up and work an example similar to Ex. 2. To Ex. 8. Ex. 10.

24. Make up and work an example similar to Ex. 14. To Ex. 18.

25. How many of the examples in this Exercise can you work at sight?

CHAPTER XXIV

THE PROGRESSIONS

274. A Series is a succession of terms formed according to some law, as

Thus, $1, 4, 9, 25, \ldots$

$1 - x + x^2 - x^3 + x^4 - , \ldots$

$2, 4, 8, 16, 32, \ldots$

275. Utility in an Algebraic Treatment of Series.

Ex. If a car going down an inclined plane travels in successive seconds, 2 ft., 6 ft., 10 ft., 14 ft., etc., how far will it go in 30 seconds?

The direct method of solution would be to set down the 30 numbers involved and add them. But by investigating the laws of the series involved and expressing these as formulas, it will be found (see Art. 278) that this long addition can be converted into two short multiplications and much labor can thus be saved.

The algebraic study of the laws of series will enable us to save labor in various ways, and to obtain other important results.

ARITHMETICAL PROGRESSION

276. An Arithmetical Progression is a series each term of which is formed by adding a constant quantity, called the *difference,* to the preceding term.

Thus, 1, 4, 7, 10, 13, · · · is an arithmetical progression (denoted by A. P.) in which the difference is 3.

Given an arithmetical progression, to determine the difference: *from any term subtract the preceding term.*

Thus, in the A. P., $\frac{3}{2}, \ -\frac{3}{4}, \ -3,$

the difference $= -\frac{3}{4} - \frac{3}{2} = -\frac{9}{4}.$

277. Quantities and Symbols. In an A. P. we are concerned with five quantities:

1. The *first term,* denoted by a.
2. The *common difference,* denoted by d.
3. The *last term,* denoted by l.
4. The *number of terms,* denoted by n.
5. The *sum of the terms,* denoted by s.

278. Two Fundamental Formulas. Since in an A. P. each term is formed by adding the common difference, d, to the preceding term, the general form of an A. P. is

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, + \ldots$$

Hence, the coefficient of d in each term is one less than the number of the term.

Thus, the 7th term is $a + 6d$,
12th term is $a + 11d$,
nth term is $a + (n - 1)d$.

Hence, $$l = a + (n - 1)d \quad \ldots \ldots \quad (1)$$

Also,

$$s = a + (a + d) + (a + 2d) + \ldots + (l - d) + l \quad \ldots \quad (2)$$

Writing the terms of this series in reverse order,

$$s = l + (l - d) + (l - 2d) + \ldots + (a + d) + a \quad \ldots \quad (3)$$

Adding (2) and (3),

$$2s = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l)$$
$$= n(a + l)$$

$$\therefore s = \frac{n}{2}(a + l) \quad \ldots \ldots \quad (4)$$

If we substitute for l in (4) from (1),

$$s = \frac{n}{2}[2a + (n - 1)d]. \quad \ldots \ldots \quad (5)$$

Hence, combining results, we have the two fundamental formulas for l and s,

$$\text{I.}\quad l = a + (n-1)d$$

$$\text{II.}\quad s = \frac{n}{2}(a + l)$$

$$s = \frac{n}{2}[2a + (n-1)d]$$

Thus formula I substitutes a multiplication for successive additions of the common difference; and formula II substitutes a multiplication for the addition of the successive terms.

Ex. 1. Find the 12th term and the sum of 12 terms of the A. P., 5, 3, 1, − 1, − 3, ...

In this series $a = 5, \quad d = -2, \quad n = 12.$

From I, $l = 5 + (12 - 1)(-2) = 5 - 22 = -17.$

From II, $s = \frac{12}{2}(5 - 17) = -72$ *Sum*

Ex. 2. Find the sum of n terms of the A. P.,

$$\frac{a+b}{2}, \quad \frac{a-b}{2}, \quad \frac{a-3b}{2}, \ldots\ldots$$

Here $a = \frac{a+b}{2}, \quad d = -b, \quad n = n.$

Substituting in the fundamental formula, $s = \frac{n}{2}[2a + (n-1)d]$,

$$s = \frac{n}{2}[a + b + (n-1)(-b)]$$

$$= \frac{n}{2}[a + (2 - n)b] \text{ \textit{Sum}}$$

EXERCISE 140

1. Give the value of d in Exs. 2–15.

2. Find the 8th term in the series 3, 7, 11, ...

3. Find the 9th term and the sum of 9 terms in 7, 3, −1, . . .

4. Find the 20th and 28th terms in 5, $\frac{13}{3}$, $\frac{11}{3}$, ...

5. Find the 16th and 25th terms in $-13\frac{1}{2}, -9, -4\frac{1}{2}, \ldots$

6. Find the 7th and 10th terms and the sum of 10 terms in the series $\frac{3}{4}, \frac{2}{3}, \frac{7}{12}, \ldots\ldots$

7. Find the 18th term and the sum of 18 terms in the series 3, 2.4, 1.8, ...

8. Find the 30th term of the series 1, 4, 7, 10, ... by successive additions of the common difference. Now find the 30th term by use of the formula. About how much shorter is the second process than the first?

Find the sum of the series:

9. 3, 8, 13, ... to 8 terms.

10. $3, -3, -9, \ldots$ to 9 terms.

11. $2\frac{1}{2}, 3\frac{3}{4}, 5, \ldots$ to 14 terms.

12. $-\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \ldots$ to 38 terms.

13. $-\frac{3}{4}, -\frac{5}{6}, -\frac{11}{12}, \ldots$ to 55 terms.

14. $5\sqrt{2}-2\sqrt{3}, 4\sqrt{2}-3\sqrt{3}, \ldots$ to 11 terms.

15. $3a-\frac{1}{a}, 2a, a+\frac{1}{a}, \ldots$ to 12 terms.

16. 1, 4, 7, 10, ... to n terms.

17. $6, 3, 0, -3, -6, \ldots$ to p terms.

18. $5, 3, 1, -1, \ldots$ to n terms.

19. $2x-y, x+y, 3y, \ldots$ to r terms.

20. Find the sum of the first 30 odd numbers by writing them down and adding them. Now find their sum by use of the formula. Compare the amount of work in the two processes.

21. How many strokes does a clock make in striking each hour of the day?

22. If a man saves \$100 in his 20th year, \$150 the next year, $200 the next, and so on through his 50th year, how much will he save in all?

23. If a body falls 16.1 ft. in one second, 3 times as far in the next second, 5 times as far in the third second, and so on, how far will it fall in 6 seconds? In 15 seconds?

24. If the velocity of a falling body at the end of 1 second is 32.2 ft. per second, at the end of the next second is 64.4 ft., at the end of the third second is 96.6 ft., what is it at the end of 10 seconds?

25. A body rolling down an inclined plane goes 6 ft. in the first second, three times as far in the next second, 5 times as far in the 3d second, and so on. How far will it go in 10 seconds?

26. State in general language the first of the formulas obtained in Art. 278.

SUG. "The last term equals the first term increased by," etc.

27. State the second formula of Art. 278 in general language.

28. State the third formula of Art. 279.

29. Make up and work an example similar to Ex. 9. To Ex. 16. Ex. 22. Ex. 25.

30. How many examples in Exercise 114 (p. 350) can you now work at sight?

279. Given Any Three of the Five Quantities *a, d, l, n, s,* to find the Other Two.

The method, in general, is as follows:

If the three known quantities are found in one of the formulas of Art. 278, substitute the three given values in the formula and solve the resulting equation;

The remaining unknown quantity may then be found by use of one of the other formulas of Art. 278;

If the three given quantities do not occur in one of the formulas of Art. 278, substitute in two of these formulas and solve by elimination.

Ex. 1. Given $l = 13$, $s = 49$, $n = 7$, find a and d.

Since the letters l, s, n, and a all occur in the formula $s = \frac{n}{2}(a + l)$, substitute the values of l, s, and n in this formula.

Hence, $49 = \frac{7}{2}(a + 13)$

$98 = 7a + 91$ $\qquad a = 1$ *Ans.*

From $l = a + (n - 1)d$ $\qquad 13 = 1 + (7 - 1)d$

whence, $d = 2$ *Ans.*

Ex. 2. Given $d = 2$, $l = 21$, $s = 121$, find a, n.

Since d, l, and s do not occur in one formula, we Substitute for d, l, s in Formulas I and II,

$$21 = a + (n - 1)2 \quad \ldots\ldots (1)$$

$$121 = \frac{n(a + 21)}{2} \quad \ldots\ldots (2)$$

$$\therefore\ a + 2n = 23 \quad \ldots\ldots (3)$$

$$an + 21n = 242 \quad \ldots\ldots (4)$$

Substitute for a in (4) from (3),

$$n(23 - 2n) + 21n = 242$$

Whence, $n = 11$ \
Hence, from (3), $a = 1$ } *Ans.*

EXERCISE 141

Find the first term and the sum of the series when

1. $d = 3$, $l = 40$, $n = 13$. **2.** $d = \frac{3}{4}$, $l = 18\frac{1}{3}$, $n = 33$.

Find the first term and the common difference when

3. $s = 275$, $l = 45$, $n = 11$. **4.** $s = 4$, $l = -10$, $n = 8$.

5. $s = -246\frac{1}{2}$, $l = -34\frac{1}{2}$, $n = 17$.

6. $s = 9$, $l = 2\frac{2}{3}$, $n = 9$.

7. $s = -\frac{47}{6}$, $l = -4$, $n = 47$.

Find n and d when

8. $a = -5, l = 15, s = 105.$

9. $a = 19, l = -21, s = -21.$

10. $a = \frac{7}{12}, l = -\frac{5}{6}, s = -2\frac{1}{4}.$

11. $a = -3\frac{1}{4}, l = 9\frac{1}{4}, s = 48.$

Find a and n when

12. $s = 10, d = 3, l = 8.$ **13.** $s = 10, d = -3, l = -4.$

14. $l = -8, d = -3, s = -3.$

15. $l = -\frac{4}{3}, d = -\frac{1}{15}, s = -\frac{40}{3}.$

How many consecutive terms must be taken in the series

16. $1, 1\frac{1}{2}, 2 \ldots.$ to make the sum 45?

17. $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \ldots.$ to make the sum -1?

18. $\frac{3}{2}, \frac{5}{4}, 1 \ldots.$ to make the sum 4.5?

19. A body rolling down an inclined plane goes 6 ft. the first second, 18 ft. the next second, 30 ft. the third second and so on. In how many seconds will it have traveled 486 ft.?

20. Make up and work an example similar to Ex. 1. To Ex. 12. Ex. 19.

21. How many examples in Exercise 45 (p. 155) can you now work at sight?

280. Arithmetical Means.

Ex. Insert 9 arithmetical means between 1 and 5.

We have given $a = 1$, $l = 5$, $n = 11$. Hence, we find $d = \frac{2}{5}$. The required means are therefore $1\frac{2}{5}, 1\frac{4}{5}, 2\frac{1}{5}, \ldots.$ *Ans.*

In case only a single arithmetical mean is to be inserted between two quantities, a and b, this one mean is found most readily by use of the formula $\frac{a+b}{2}$. For if x denotes the required mean, the A. P. is a, x, b.

$$\begin{aligned} \text{Hence,}\quad x - a &= b - x \\ 2x &= a + b \\ x &= \frac{a+b}{2} \end{aligned}$$

EXERCISE 142

Insert

. Four arithmetical means between 7 and -3.

. Seven arithmetical means between 4 and 6.

Thirteen arithmetical means between $\frac{1}{3}$ and $-\frac{5}{6}$.

4. Fifteen arithmetical means between $-4\frac{1}{3}$ and 9.

5. The arithmetical mean between $2\frac{1}{5}$ and $-5\frac{2}{3}$.

6. The arithmetical mean between $x + 1$ and $x - 1$.

7. Find the A. M. between $\frac{a}{b}$ and $\frac{b}{a}\cdot$ $\frac{1}{x+y}$ and $\frac{1}{x-y}\cdot$

8. If the height of Bunker Hill Monument is 221 ft., of the Washington Monument 555 ft., and the length of the Olympic 882 ft., by how much does the middle one of these numbers differ from the arithmetical mean between the other two?

9. Make up and work a similar example concerning $12\frac{1}{4}$ mi., 49 mi., and 100 mi., which are the lengths of the Simplon Tunnel, the Panama Canal, and the Suez Canal respectively.

10. Rome was founded 753 B. C. and fell 476 A. D. How far is the latter number from being an arithmetical mean between the former and the number of the year in which Columbus discovered America?

11. Ether boils at a temperature of 96° F., alcohol at 167°, and water at 212°. How far is 167° from being an arithmetical mean between the other two temperatures?

12. Show that if twice one number equals the sum of two other numbers, the three numbers may be arranged as an A. P.

13. State the formula $x = \frac{a+b}{2}$ (of Art. 280) in general language.

14. Work at sight such examples on pp. 101–102 as the teacher may indicate.

281. Miscellaneous Examples.

Ex. 1. The 7th term of an A. P. is 5, and the 14th term is -9. Find the first term.

By the use of Formula I (Art. 278),

the 7th term is $a + 6d$, and the 14th term is $a + 13d$.

$$\therefore a + 6d = 5 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$a + 13d = -9 \quad \ldots\ldots\ldots\ldots \quad (2)$$

Subtracting (1) from (2), $7d = -14$

$$d = -2$$

Substitute for d in (1), $a - 12 = 5$

$$a = 17 \ \textit{Ans.}$$

Ex. 2. The sum of five numbers in A. P. is 15, and the sum of the 1st and 4th numbers is 9. Find the numbers.

Denote the numbers by

$$x - 2y, \quad x - y, \quad x, \quad x + y, \quad x + 2y$$

Add, $5x = 15 \quad \ldots\ldots\ldots\ldots \quad (1)$

Also, $(x - 2y) + (x + y) = 9$

$$\therefore 2x - y = 9 \quad \ldots\ldots\ldots\ldots \quad (2)$$

From (1) $x = 3$; hence, from (2), $y = -3$.

Hence, the numbers are 9, 6, 3, 1, -3, *Ans.*

Similarly, in dealing with four unknown quantities in A. P., we denote them by

$$x - 3y, \quad x - y, \quad x + y, \quad x + 3y$$

EXERCISE 143

Find the first two terms of the series wherein

1. The 4th term is 11 and the 10th is 23.

2. The 6th term is -3 and the 12th is -12.

3. The 7th term is $-\frac{1}{6}$ and the 16th is $2\frac{1}{3}$.

4. The 5th term is $c - 3b$ and the 11th is $3b - 5c$.

5. Find the sum of the first n odd numbers. State the result obtained as a rule in general language.

6. Set down the first 20 odd numbers and find their sum by addition. Now find their sum by the formula result obtained in the preceding example. Compare the amount of work in the two processes.

7. Find the sum of the first n numbers divisible by 7.

8. Make up and work an example similar to Ex. 6, but showing the utility of the result obtained in Ex. 7.

9. Which term in the series $1\frac{1}{6}$, $1\frac{1}{3}$, $1\frac{1}{2}$, $\cdots$ is 18?

10. The first term of an arithmetical progression is 8; the 3d term is to the 7th as the 8th is to the 10th. Find the series.

11. Find four numbers in A. P., such that the sum of the first two is 1, and the sum of the last two is -19.

12. Find four numbers in A. P. whose sum is 16 and product is 105.

13. A man travels $2\frac{1}{3}$ mi. the first day, $2\frac{2}{3}$ the second, 3 the third, and so on; at the end of his journey he finds that if he had traveled $6\frac{1}{6}$ mi. every day he would have required the same time. How many days was he walking?

14. The sum of 10 numbers in an A. P. is 145, and the sum of the fourth and ninth terms is 5 times the third term. Find the series.

15. If the 11th term is 7 and the 21st term is $8\frac{2}{3}$, find the 41st term of the same A. P.

16. In an A. P. of 21 terms the sum of the last three terms is 23, and the sum of the middle three is 5. Find the series.

17. Required five numbers in A. P., such that the sum of the first, third, and fourth terms shall be 8, and the product of the second and fifth shall be -54.

18. The sum of five numbers in A. P. is 40, and the sum of their squares is 410. Find them.

19. The 14th term of an A. P. is 38; the 90th term is 152, and the last term is 218. Find the number of terms.

20. How many numbers of two figures are there divisible by 3? By 7? How many numbers of three figures are divisible by 6? By 9?

21. How many numbers of four figures are there divisible by 11? Find the sum of all the numbers of three figures divisible by 7.

22. If a car starts at the top of a hill and runs down an inclined track 2 ft. the first second, 6 ft. the next second, 10 ft. the next, etc., and reaches the bottom in 12 seconds, how long is the track?

23. Sulphur fuses at a temperature of 239° F., tin at 442°, and lead at 617°. By how much does 442° differ from the arithmetical mean between the other two temperatures?

24. Copper fuses at a temperature of 2200° F., gold at 2518°, and iron at 2800°. Treat these temperatures in a way similar to that used in the preceding example.

25. The heights of Mt. Washington, Pike's Peak, Mt. McKinley, and Mt. Everest are respectively 6290 ft., 14,147 ft., 20,464 ft., and 29,002 ft. Find the difference between each of these numbers and the corresponding term in an A. P. whose first term is 6290 ft. and common difference 7500 ft.?

26. In an A. P. whose second term is 14,200 ft. and common difference 7600 ft.?

27. If a body falls $16\frac{1}{12}$ ft. in the first second; three times this distance in the next; five times in the third, and so on, how far will it fall in the 30th second? How far will it have fallen during the 30 seconds? In how many seconds will it have fallen $6433\frac{1}{3}$ ft.?

28. If $a, b, c, d,$ are in A. P. prove: (1) that $a + d = b + c$; (2) that ak, bk, ck, dk are also in A. P.; and (3) that $a + k$, $b + k$, $c + k$, $d + k$ are in A. P. State this problem without the use of the symbols, a, b, c, d, k.

29. Make up and work an example similar to Ex. 1. To Ex. 7. Ex. 22.

30. Practice oral work with fractions as in Exercise 58 (p. 190).

GEOMETRICAL PROGRESSION

282. **A Geometrical Progression** is a series each term of which is formed by multiplying the preceding term by a constant quantity called the *ratio*.

Thus, 1, 3, 9, 27, 81, ... is a geometrical progression (or G. P.) in which the ratio is 3.

Given a geometrical progression, to determine the ratio: *divide any term by the preceding term.*

Thus, in the G. P., $-3, \frac{3}{2}, -\frac{3}{4}, \ldots\ldots$

the ratio $= \dfrac{\frac{3}{2}}{-3} = -\dfrac{1}{2}$

283. Quantities and Symbols. The symbols a, l, n, s are used, as in A. P. Besides these, r is used to denote the ratio.

284. Two Fundamental Formulas. Since in a G. P. each term is formed by multiplying the preceding term by the common ration, r, the general form of a G. P. is

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \ldots\ldots$$

Hence, the exponent of r in each term is one less than the number of the term.

Thus, the 10th term is ar^9.

15th term is ar^{14}.

nth term, or $l = ar^{n-1}$ . . . . (1)

In deriving a formula for the sum, we know, also,

$$s = a + ar + ar^2 + \ldots\ldots + ar^{n-1} \quad \ldots\ldots (2)$$

Multiply (2) by r,

$$rs = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n. \quad . (3)$$

Subtract (2) from (3),

$$rs - s = ar^n - a$$

$$\therefore s = \frac{ar^n - a}{r - 1} \quad \ldots\ldots\ldots\ldots (4)$$

Multiply (1) by r, $rl = ar^n$

Substitute rl for ar^n in (4),

$$s = \frac{rl - a}{r - 1} \quad \ldots\ldots\ldots\ldots (5)$$

Hence, collecting the results obtained in (1), (4), (5), we have the two fundamental formulas for l and s:

$$\text{I.} \quad l = ar^{n-1}$$

$$\text{II.} \quad s = \frac{ar^n - a}{r - 1}$$

$$s = \frac{rl - a}{r - 1}$$

Ex. 1. Find the 8th term and sum of 8 terms of the G. P., 1, 3, 9, 27, ...

In this case, $a = 1, \quad r = 3, \quad n = 8$

From I, $l = 1 \times 3^7 = 2187$

From II, $s = \frac{3 \times 2187 - 1}{3 - 1} = 3280$ *Sum*

Ex. 2. Find the 10th term and the sum of 10 terms of the G. P., $4, -2, 1, -\frac{1}{2}$, ...

Here $a = 4, \quad r = -\frac{1}{2}, \quad n = 10$

Hence, $l = 4(-\frac{1}{2})^9 = -\frac{4}{512} = -\frac{1}{128}$

$s = \frac{(-\frac{1}{2})(-\frac{1}{128}) - 4}{-\frac{1}{2} - 1} = \frac{341}{128}$ *Sum*

EXERCISE 144

Give the value of r in Exs. 2–15.

1. Find the 6th term in the series 2, 6, 18, ...

2. Find the 7th term in 3, 6, 12, ...

3. Find the 6th and the sum of 6 terms in 45, −15, 5, . . .

4. Find the 5th and the sum of 5 terms in 81, −54, ...

5. Find the 7th and the sum of 7 terms in $1\frac{1}{8}, -\frac{3}{4}$, ...

6. Find the 9th term in the series $2, 2\sqrt{2}, 4$, ...

7. 15th term of $\frac{1}{b}, \frac{1}{b^2}, \frac{1}{b^3}$, ...

8. nth term of $p, \frac{p}{q}, \frac{p}{q^2}, \frac{p}{q^3}$, ...

Find the sum of the series

9. 3, − 6, 12, ... to 6 terms.

10. 27, − 18, 12, ... to 7 terms.

11. $-\frac{8}{9}, 1\frac{1}{3}, -2$, ... to 9 terms.

12. $\frac{2}{3}, -\frac{1}{6}, \frac{1}{24}$, ... to 8 terms.

13. $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, \ldots.$ to 8 terms.

14. $\sqrt{2}-1, 1, \sqrt{2}+1, \ldots.$ to 6 terms.

15. 1, 2, 4, 8, ... to n terms.

16. The following is a series of specific gravities: cork, .25; oak wood, .75; aluminum, 2.5; iron, 7.5; platinum, 21.5. By how much does each term of this series differ from the corresponding term in a G. P. whose first term is .25 and whose ratio is 3?

(What is meant by specific gravity?)

17. If the average age of parents be taken as 30 years, find the total number of a person's ancestors in a period of 600 years.

18. The population of the United States in the year 1900 was 76,300,000. If this should increase 50% every 25 years, what would the population be in the year 2000?

19. If a man saves $300 each year for 10 years, what is the amount of his savings in 5 years at compound interest at 5 per cent? In 10 years?

20. A ship was built at a cost of $70,000. Her owners at the end of each year deducted 10% from her value as estimated at the beginning of the year. What is her estimated value at the end of 10 years?

21. A grain of wheat when planted produced a stalk on which were 30 other grains. The next year each of the grains was planted and produced similar stalks. If this process were continued, at the end of 10 years how many bushels would be produced in the last crop if 1 quart contains 2000 grains?

22. Make up and work a similar example concerning the amount of corn produced from one grain, using probable numbers.

23. If 32 nails are used in shoeing a horse, make up and work an example concerning a man who paid a blacksmith for shoeing a horse at the rate of $\frac{1}{4}$¢ for the first nail driven, $\frac{1}{2}$¢ for the second nail, 1¢ for the third, etc.

State in general language:

24. The first formula obtained in Art. 284.

25. The second formula. The third formula.

26. Make up and work an example similar to Ex. 4. To Ex. 15. Ex. 20.

27. Practice the oral solution of simple equations as in Exercise 64 (p. 209).

285. Given Three of the Five Quantities *a, l, n, s, r*, to determine the Other Two.

Use the same general method as that given in Art. 279 (p. 427), for A. P.

Ex. 1. Given $a = -2$, $n = 7$, $l = -128$; find r, s.

From I, $\quad -128 = -2r^6$

Hence, $\quad r^6 = 64, \quad r = \pm 2$

From II, if $r = +2$, $s = \dfrac{2(-128)\ -_{1}\,(-2)}{2\ -} = -256 + 2 = -254$

If $r = -2$, $\quad s = \dfrac{(-2)(-128)-(-2)}{-2-1} = \dfrac{256+2}{-3} = -86$

Hence, there are two sets of answers; viz.,

$$\left.\begin{array}{l} r = +2, \quad s = -254_{9} \\ r = -2, \quad s = -86 \end{array}\right\} \textit{Ans.}$$

Ex. 2. Given, $a = \frac{3}{4}$, $r = -\frac{1}{3}$, $s = \frac{1}{162}$; find l, n.

The most convenient method of solution is to find, first l, then n.

Substituting in the formula $s = \dfrac{rl - a}{r - 1}$

$$\frac{91}{162} = \frac{-\frac{1}{3}l - \frac{3}{4}}{-\frac{3}{4} - 1}, \text{ whence } l = -\frac{1}{324} \textit{ Ans.}$$

Using $l = ar^{n-1}$, $-\frac{1}{324} = \frac{3}{4}(-\frac{1}{3})^{n-1}$

Whence, $-\frac{1}{324} \times \frac{4}{3} = (-\frac{1}{3})^{n-1}$, and $n = 6$ *Ans.*

Let the pupil check these results.

EXERCISE 145

Find the first term and the sum when

1. $n = 6, r = 3, l = 486.$ **2.** $n = 8, r = -2, l = -640.$

3. $n = 8, r = -\frac{2}{3}, l = -\frac{640}{243}.$

4. $n = 7, r = \frac{1}{2}\sqrt{6}, l = 3.$

Find the ratio when

5. $a = -2, l = 2048, n = 6.$

6. $a = 9, l = \frac{16}{9}, s = 23\frac{4}{9}.$

7. $a = 2\frac{26}{27}, l = -\frac{45}{64}, n = 6.$

8. $a = -16\frac{1}{5}, l = \frac{1}{15}, s = -12\frac{2}{15}.$

Find the number of terms when

9. $a = 2, r = 2, s = 62.$ **11.** $a = \frac{4}{3}, l = \frac{1}{12}, r = \frac{1}{2}.$

10. $a = 4, r = -\frac{1}{2}, s = 2\frac{3}{4}.$ **12.** $a = 3, l = -96, s = -63.$

13. $a = 18, r = -\frac{2}{3}, s = 12\frac{2}{9}.$

How many consecutive terms must be taken from the series

14. 2, 4, 8, ... to make the sum 62?

15. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}$, ... to make the sum $\frac{211}{108}$?

16. $5\frac{1}{3}, -8, 12$, ... to make $-22\frac{1}{6}$?

17. Make up and work an example similar to Ex. 1. To Ex. 13.

18. How many examples in Exercise 83 (p. 273) can you now work at sight?

286. Geometric Means.

Ex. Insert 5 geometrical means between 3 and $\frac{1}{243}$.

We have given $a = 3$, $l = \frac{1}{243}$, $n = 7$, to find r.
Solving by Art. 285, $r = \frac{1}{3}$
Hence, the required geometrical means are,

$$1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \tfrac{1}{27},\ \tfrac{1}{81}\ \textit{Ans.}$$

In case only one geometrical mean is to be inserted between two quantities, a and b, this one mean is found most readily by using the formula $\sqrt{ab}$. For if x represents the geometrical mean between a and b, the series will be

$$a,\quad x,\quad b$$

Hence, $\frac{x}{a} = \frac{b}{x}$, $\therefore x^2 = ab$, $x = \sqrt{ab}$

EXERCISE 146

Insert

1. Three geometrical means between 8 and $\frac{1}{2}$.
2. Three geometrical means between $\frac{9}{4}$ and $\frac{4}{9}$.
3. Six geometrical means between $\frac{1}{24}$ and $-\frac{16}{3}$.
4. Four geometrical means between $-\frac{7}{2}$ and 3584.
5. Six geometrical means between 56 and $-\frac{7}{16}$.

Find the geometrical mean between

6. $4\frac{1}{6}$ and $\frac{3}{8}$.
7. $3\frac{8}{9}$ and $6\frac{3}{7}$.
8. $28a^3x$ and $63axy^4$.
9. $\dfrac{a\sqrt{x}}{c^2\sqrt[3]{y}}$ and $\dfrac{y\sqrt[3]{a^2}}{x\sqrt{c^3}}$.
10. .7 and .343.
11. .5 and .125.
12. .005 and .125.
13. $5\sqrt{2}+1$ and $5\sqrt{2}-1$.
14. Insert 6 geometrical means between $\dfrac{a^3}{16}$ and $\dfrac{8}{\sqrt{a}}$.

15. Insert 7 geometrical means between $\frac{8}{n^2}$ and $\frac{n^2}{2}$.

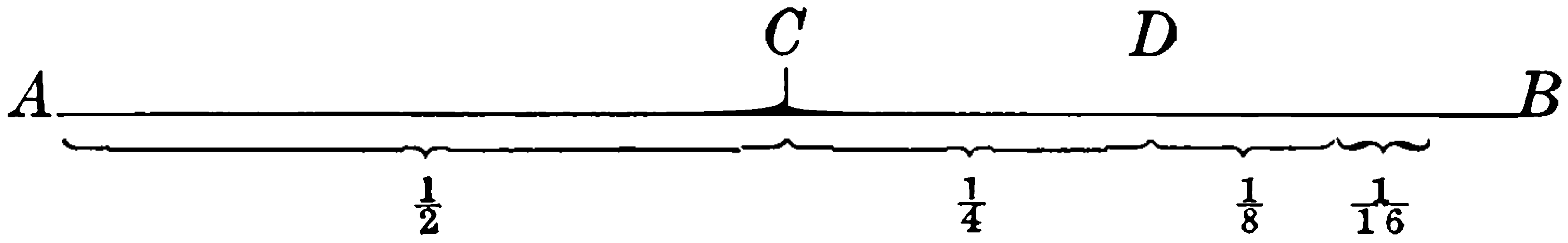
16. Is a mean proportional between two numbers the same as the geometric mean between the numbers?

17. State the formula $x = \sqrt{ab}$ (of Art. 286) in general language.

18. Make up and work an example similar to Ex. 2. To Ex. 12.

19. How many examples in Exercise 85 (p. 280) can you now work at sight?

287. Limit of the Sum of an Infinite Decreasing Geometrical Progression.



If a line *AB* is of unit length, and one half of it (*AC*) is taken, and then one half of the remainder (*CD*), and one half of the remainder, and so on, the sum of the parts taken will be

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \dots\dots$$

This is an infinite decreasing G. P. in which $r = \frac{1}{2}$. The sum of all these parts must be less than 1, but must approach closer and closer to 1 as a limit, the greater the number of parts taken. This illustrates the meaning of the limit of an infinite decreasing G. P.

In general, to find the limit of an infinite decreasing G. P. we have the formula

$$s = \frac{a}{1 - r} \quad \dots\dots\dots\dots \text{III}$$

Formula II of Art. 284 may be written, $s = \frac{a - rl}{}$.

Then, as the number of terms increases,

$$\begin{array}{rlcl} & l & \text{approaches indefinitely to} & 0 \\ \therefore & rl & \text{“} \qquad \text{“} & 0 \\ \therefore & a - rl & \text{“} & a - 0 = a \\ \therefore & \dfrac{a - rl}{1 - r} & \text{“} & \dfrac{a}{1 - r} \end{array}$$

$$\therefore s = \frac{a}{1 - r}$$

Ex. Find the sum of 9, -3, 1, $-\frac{1}{3}$, ... to infinity.

Here $a = 9, \quad r = -\frac{1}{3}$.

$$\therefore s = \frac{9}{1 - (-\frac{1}{3})} = \frac{9}{1 + \frac{1}{3}} = \frac{27}{4} \textit{ Sum}$$

288. Repeating Decimals. By the use of Art. 287, the value of repeating decimals may be determined.

Ex. 1. Find the value of .373737 . . . . .

$$.373737 \ldots\ldots = .37 + .0037 + .000037 + \ldots\ldots$$

Here $a = .37, \quad r = .01$

$$\therefore s = \frac{.37}{1 - .01} = \frac{.37}{.99} = \frac{37}{99} \textit{ Ans.}$$

Ex. 2. Find the value of 3.1186186 . . . . .

Setting aside 3.1, and treating the remaining terms as a G. P.,

$$a = .0186, \qquad r = .001$$

$$\therefore s = \frac{.0186}{1 - .001} = \frac{.0186}{.999} = \frac{186}{9990} = \frac{62}{3330}$$

$$\therefore 3.1186186 \ldots\ldots = 3\tfrac{1}{10} + \tfrac{62}{3330} = 3\tfrac{79}{666} \textit{ Ans.}$$

EXERCISE 147

Find the sum to infinity of the series

1. $2, \frac{2}{3}, \frac{2}{9}, \ldots\ldots$
2. $2, -1, \frac{1}{2}, \ldots\ldots$
3. $-9, 6, -4, \ldots\ldots$
4. $1\frac{7}{8}, 1\frac{1}{4}, \frac{5}{6}, \ldots\ldots$
5. $4\frac{1}{6}, -2\frac{1}{2}, 1\frac{1}{2}, \ldots\ldots$
6. $\frac{1}{12}, \frac{1}{18}, \frac{1}{27}, \ldots\ldots$
7. $2\frac{13}{16}, -1\frac{7}{8}, 1\frac{1}{4}, \ldots\ldots$
8. $6, 3\sqrt{2}, 3, \ldots\ldots$
9. $\dfrac{1}{\sqrt{2} - 1}, 1, \dfrac{1}{\sqrt{2} + 1}, \ldots\ldots$

10. $\frac{1}{2}\sqrt{2} + \frac{1}{3}\sqrt{3} + \frac{1}{3}\sqrt{2}, \ldots\ldots$

11. $1 + \left(1 - \frac{1}{a}\right) + \left(1 - \frac{1}{a}\right)^2 + \ldots\ldots$

12. Give the ratio in the G. P. in each of the following: (1) .333 .... (2) .272727 .... (3) .356356. (4) .79127912. (5) .5333 .... (6) In Exs. 13–21.

Find the values of

13. 0.63 **14.** 0.417 **15.** 5.846

16. 3.52424 ..... **19.** 1.02727 .....

17. 1.4037037 ..... **20.** 1.027027 .....

18. 3.215454 ..... **21.** .30102102 .....

22. Find the first term in an infinite decreasing geometrical progression whose sum is $\frac{3}{2}$ and whose ratio is $-\frac{1}{3}$.

23. If the velocity of a sled at the foot of a hill is 60 ft. per second and this velocity should be diminished by one third each second as the sled moves out on the horizontal, how far would the sled move before coming to rest?

24. Make up and solve a similar example concerning a car which ran down an inclined track out on a horizontal track.

25. If a ball, dropped from a height of 80 ft., rebounded 40 ft.; and, on striking the ground again, rebounded 20 ft., and so on, how far would it travel before coming to rest?

26. Make up and work an example similar to Ex. 25.

27. State the formula $s = \frac{a}{1-r}$ in general language.

28. Make up and work an example similar to Ex. 3. To Ex. 16.

29. Practice oral work with exponents as in Exercise 93 (p. 303).

289. Miscellaneous Problems.

Ex. Find four members in G. P., such that the sum of the first and fourth is 56, and of the second and third is 24.

Denote the required numbers by a, ar, ar^2, ar^3.

$$\begin{array}{ll}
\text{Then} & a + ar^3 = 56 \\
 & ar + ar^2 = 24 \\
\text{Or,} & a(1 + r^3) = 56 \quad \ldots\ldots\ldots\ldots (1) \\
 & ar(1 + r) = 24 \quad \ldots\ldots\ldots\ldots (2) \\
\text{Divide (1) by (2),} & \dfrac{1 - r + r^2}{r} = \dfrac{7}{3} \\
\text{Hence,} & 3 - 3r + 3r^2 = 7r \\
 & 3r^2 - 10r = -3 \\
 & r = 3, \quad \text{or } \tfrac{1}{3} \\
\text{And} & a = 2, \quad \text{or } 54
\end{array}$$

$$\begin{array}{ll}
\text{Hence, the numbers are,} & \left.\begin{array}{l} 2,\ 6,\ 18,\ 54 \\ 54,\ 18,\ 6,\ 2 \end{array}\right\} \ \mathit{Ans.} \\
\text{Or,} &
\end{array}$$

EXERCISE 148

Find the first two terms of the series in which

1. The 3d term is 2, and the 5th is 18.

2. The 4th term is $\frac{3}{2}$ and the 9th is 48.

3. The 5th term is 6 and the 11th is $\frac{3}{32}$.

Determine the nature, whether Arithmetic or Geometric, of each of the following series:

4. $\frac{1}{4}, \frac{1}{3}, \frac{4}{9}, \ldots\ldots$

5. $\frac{1}{4}, \frac{1}{6}, \frac{1}{9}, \ldots\ldots$

6. $\frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \ldots\ldots$

7. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \ldots\ldots$

8. $3\frac{1}{5}, 4\frac{4}{5}, 7\frac{1}{5}, \ldots\ldots$

9. $7\frac{1}{3}, 5\frac{7}{8}, 4\frac{5}{12}, \ldots\ldots$

10. Divide 65 into 3 parts in geometrical progression, such that the sum of the first and third is $3\frac{1}{3}$ times the second part.

11. There are 3 numbers in G. P. whose sum is 49, and the sum of the first and second is to the sum of the first and third as 3 to 5. Find them.

12. The sum of three numbers in G. P. is 21, and the sum of their reciprocals is $\frac{7}{12}$. Find the numbers.

13. Find four numbers in G. P., such that the sum of the first and third is 10, and of the second and fourth 30.

14. Three numbers whose sum is 24 are in A. P., but if 3, 4, and 7 be added to them respectively, these sums will be in G. P. Find the numbers.

15. The sum of $225 was divided among four persons in such a manner that the shares were in G. P., and the difference between the greatest and least was to the difference between the means as 7 is to 2. Find each share.

16. Find the sum of $\frac{1}{\sqrt{2}-1}$, $\sqrt{2}$, $\frac{2}{\sqrt{2}+1}$ . . . *ad infinitum.*

17. There are four numbers the first three of which are in G. P., and the last three are in A. P.; the sum of the first and last is 14, and of the means is 12. Find the numbers.

18. If the series $\frac{3}{4}$, $\frac{1}{2}$ . . . . be arithmetical, find the 102d term; if geometrical, find the sum to infinity.

19. Insert between 2 and 9 two numbers, such that the first three of the four may be in A. P., and the last three in G. P.

20. Prove that the series $\sqrt{2}-1$, $3\sqrt{2}-4$, $2(5\sqrt{2}-7)$ . . . . is geometrical; that its ratio is $2-\sqrt{2}$; and that its sum to infinity is unity.

21. The cost per ounce of mailing different kinds of mail matter is given in the following series: 2¢, 1¢, $\frac{1}{2}$¢, $\frac{1}{4}$¢, $\frac{1}{16}$¢. How far does each of these numbers differ from the corresponding term in a G. P. whose first term is 2¢ and whose ratio is $\frac{1}{2}$? (Name the class of mail matter and some of the principal articles to which each of the above rates belongs.)

22. If the areas of Rhode Island, New Jersey, New York, and Texas are respectively 1250, 7815, 49,170, and 265,780 sq. mi., how far are these numbers from forming a G. P. of which the second term is 7815 and the ratio 6?

23. On p. 395 a table gives the amount of $1 in different periods of time at simple interest and also at compound interest. Which of the two series of numbers forms an A. P. and which a G. P.?

24. If an air pump at each stroke removes $\frac{1}{8}$ the air in a receiver, what fraction of the air is left at the end of 10 strokes?

25. If the amount of air in a receiver is indicated by the height of a mercury column in a tube attached to the receiver, and this height is 30 in. at the start, what will the height of the mercury be at the end of the 10 strokes?

26. There were 2,500,000,000,000 tons of coal in the United States in the year 1910, and 3,000,000,000 tons were consumed between the year 1900 and 1910. If the consumption of coal should double every decade, tell to the nearest decade how long the coal in the United States would last?

27. Work again Exercise 76 (p. 249), or similar examples suggested by the teacher or pupils.

CHAPTER XXV

THE BINOMIAL THEOREM

FOR POSITIVE INTEGRAL EXPONENTS

290. The Binomial Formula. The results obtained by inspection in Art. 166 (p. 276) may be combined in a formula as follows:

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{\underline{|2}}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}a^3 + \cdots\cdot$$

We shall now give a proof of this formula for all positive integral values of n.

291. Proof of the Binomial Formula for Positive Integral Values of *n*. This proof may be conveniently divided into three parts.

I. By actual multiplication it is found that, for any definite value of n, as $n = 4$,

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4.$$

That is, the binomial formula is true when $n = 4$.

II. We shall now prove the general principle that if the binomial formula is true for any power, as the kth, it is true for the next higher power, the $k + 1$ power.

We write out the formula for the kth power and multiply both sides by $x + a$.

$$
\begin{array}{l}
(x+a)^k = x^k + kx^{k-1}a + \frac{k(k-1)}{1\times 2}x^{k-2}a^2 + \frac{k(k-1)(k-2)}{1\times 2\times 3}x^{k-3}a^3 + \cdots \\
x + a \quad , \quad x + a \\
\hline
(x+a)^{k+1} = x^{k+1} + kx^k a + \frac{k(k-1)}{1\times 2}x^{k-1}a^2 + \frac{k(k-1)(k-2)}{1\times 2\times 3}x^{k-2}a^3 + \cdots \\
\qquad\qquad\qquad + x^k a \qquad\qquad + kx^{k-1}a^2 \qquad\quad + \frac{k(k-1)}{1\times 2}x^{k-2}a^3 + \cdots \\
\hline
(x+a)^{k+1} = x^{k+1} + (k+1)x^k a + \left[\frac{k(k-1)}{1\times 2} + k\right]x^{k-1}a^2 \\
\qquad\qquad\qquad + \left[\frac{k(k-1)(k-2)}{1\times 2\times 3} + \frac{k(k-1)}{1\times 2}\right]x^{k-2}a^3 + \cdots
\end{array}
$$

or, $(x+a)^{k+1}$

$$= x^{k+1} + (k+1)x^k a + \frac{(k+1)k}{1\times 2}x^{k-1}a^2 + \frac{(k+1)k(k-1)}{1\times 2\times 3}x^{k-2}a^3 + \cdots$$

This is the result which would be obtained by expanding $(x+a)^{k+1}$ by the formula.

Hence, we have proved that if the binomial formula is true for any power, as the kth, it is true for the next higher power, the $k+1$ power.

III. But by actual multiplication (in I) the binomial formula was shown to be true for $(x+a)^4$, or the 4th power. Hence by the general principle just proved (in II), the formula must be true for the next higher power, the 5th. In like manner, it must be true for the 6th, etc., to the nth power.

The method of proof used in this Article is called *mathematical induction.*

292. When a is negative, a^3, a^5, etc., are negative; hence,

$$
\begin{array}{l}
(x-a)^n = x^n - nx^{n-1}a + \frac{n(n-1)}{1\times 2}x^{n-2}a^2 \\
\qquad\qquad - \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}a^3 + \cdots\cdots \quad (1)
\end{array}
$$

This formula may be proved by changing a into $-a$ in the proof given in Art. 291.

EXERCISE 149

1. Write out the formula for $(x+a)^7$.

2. For $(x+a)^{n+1}$. **3.** For $(x+a)^{n-1}$.

4. How many terms are there in the expansion of $(x+a)^2$? Of $(x+a)^3$? $(x+a)^{10}$? $(x+a)^n$? $(x+a)^{n-2}$?

Before the pupil attempts the proof of the following laws, each law should be illustrated numerically till its meaning is thoroughly understood.

5. By mathematical induction prove that

$$1+2+3+\ldots\ldots+n=\frac{n}{2}(n+1)$$

SUG. (1) We have $1+2+3=\frac{3}{2}(3+1)$, or 6

(2) If $1+2+3+\ldots\ldots+k=\frac{k}{2}(k+1)$

adding $k+1$ to each member,

$$1+2+3+\ldots\ldots+k+(k+1)=\frac{k}{2}(k+1)+k+1$$

$$=\frac{(k+1)(k+2)}{2}$$

(3) Hence, etc.

By mathematical induction, prove that

6. The sum of the first n even numbers equals $n(n+1)$.

7. The sum of the first n odd numbers equals n^2.

8. $1^2+2^2+3^2+\ldots\ldots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

9. $2^2+4^2+6^2+\ldots\ldots+(2n)^2=\frac{2}{3}n(n+1)(2n+1)$.

10. $1^3+2^3+3^3+\ldots\ldots+n^3=\frac{1}{4}n^2(n+1)^2$

$$=(1+2+3+\ldots\ldots+n)^2.$$

11. a^n-b^n is always divisible by $a-b$ when n is an integer.

12. Make up and work two examples to be solved by the method of mathematical induction.

293. Key Number and rth Term. In memorizing the binomial formula, it is helpful to observe that a certain number may be regarded as governing the formation of each term of the formula. This number is one less than the number of the term.

Thus, for the *third* term we have $\frac{n(n-1)}{1 \times 2}x^{n-2}a^2$, in which there are *two* factors in the numerator of the coefficient; *two* in the denominator; the exponent of x is $n-2$, and that of a is 2. Hence, we regard 2 as the *key number* of the term.

The number 3 occurs in a similar way in the formation of the fourth term; 4, in the fifth term, and so on.

For the rth term, the key number would be $r-1$.

Hence,

$$r\text{th term} = \frac{n(n-1)\ \ldots\ldots\ (n-r+2)}{\left|\underline{r-1}\right.}x^{n-r+1}a^{r-1}$$

294. Examples.

Ex. 1. Expand $\left(\frac{x^2}{2} - \frac{1}{\sqrt{x^2}}\right)^6$.

$$\left(\frac{x^2}{2} - \frac{1}{\sqrt[3]{x^2}}\right)^6 = \left(\frac{x^2}{2} - x^{-\frac{2}{3}}\right)^6$$

$$= \left(\frac{x^2}{2}\right)^6 - 6\left(\frac{x^2}{2}\right)^5\left(x^{-\frac{2}{3}}\right) + 15\left(\frac{x^2}{2}\right)^4\left(x^{-\frac{2}{3}}\right)^2 - 20\left(\frac{x^2}{2}\right)^3\left(x^{-\frac{2}{3}}\right)^3$$

$$+ 15\left(\frac{x^2}{2}\right)^2\left(x^{-\frac{2}{3}}\right)^4 - 6\left(\frac{x^2}{2}\right)\left(x^{-\frac{2}{3}}\right)^5 + \left(x^{-\frac{2}{3}}\right)^6$$

$$= \frac{x^{12}}{64} - \frac{3x^{\frac{28}{3}}}{16} + \frac{15x^{\frac{20}{3}}}{16} - \frac{5x^4}{2} + \frac{15x^{\frac{4}{3}}}{4} - 3x^{-\frac{4}{3}} + x^{-4} \quad \textit{Ans}$$

Ex. 2. Find the sixth term of $\left(\frac{x}{2} - \frac{2}{3\sqrt{xy}}\right)^9$.

The key number for the sixth term is 5. Hence we obtain

$$\text{Sixth term} = \frac{9 \times 8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4 \times 5}\left(\frac{x}{2}\right)^{9-5}\left(-\frac{2}{3}x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)^5$$

$$= -\frac{126}{1} \cdot \frac{x^4}{2^4} \cdot \frac{2^5x^{-\frac{5}{2}}y^{-\frac{5}{2}}}{3^5} = -\frac{28}{27}x^{\frac{3}{2}}y^{-\frac{5}{2}} \quad \textit{Ans.}$$

Ex. 3. Find the term in $\left(x^2 - \dfrac{2}{x^{\frac{1}{2}}}\right)^{11}$ which contains x^{12}.

We must first find the number of the term and then the term itself.

The rth term of $(x^2 - 2x^{-\frac{1}{2}})^{11} = (\text{coeff.})\ (x^2)^{11-r+1}\ (-2x^{-\frac{1}{2}})^{r-1}$.
For the required term, the x's collected must $= x^{12}$.

Hence, $(x^2)^{12-r}(x^{-\frac{1}{2}})^{r-1} = x^{12}$

$$24 - 2r - \frac{r-1}{2} = 12$$

Whence, $r = 5$

5th term $= \dfrac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4}(x^2)^{11-4}\ (-2x^{-\frac{1}{2}})^4$

$= 5280x^{12}$ *Ans.*

EXERCISE 150

1. Change each of the given expressions in Exs. 6–15 to a form in which it can be most readily expanded by the binomial formula.

Expand:

2. $(2a - x)^5$.

3. $\left(1 + \dfrac{x}{2}\right)^6$.

4. $(3x^{\frac{1}{2}} - 2y^2)^4$.

5. $(x^{\frac{1}{3}} - 2x)^5$.

6. $(x^{-\frac{2}{3}} + \sqrt{x})^7$.

7. $\left(\dfrac{x}{2y} - \sqrt{xy}\right)^5$.

8. $\left(x^{-2} - \dfrac{y^3}{3}\right)^5$.

9. $\left(3\sqrt{\dfrac{a}{x^3}} - 1\right)^5$.

10. $(2\sqrt{x} + \sqrt[3]{xy^2})^4$.

11. $\left(\dfrac{2x}{\sqrt{y}} - \dfrac{y}{2\sqrt{x}}\right)^5$.

12. $\left(2\sqrt[3]{\dfrac{a^2}{x}} + 3\sqrt{\dfrac{x}{a^3}}\right)^6$.

13. $(3a^{-\frac{3}{4}}\sqrt{b} - b^{-\frac{7}{2}}\sqrt[4]{a})^4$.

14. $(x^2 - x + 2)^3$.

15. $(2 - 3x + x^2)^3$.

16. $(2x^2 + x - 3)^4$.

17. $(a^2 + 2ax - x^2)^4$.

Find the

18. Sixth term of $(a - 2x^2)^{11}$.

19. Eighth term of $(1 + x\sqrt{y})^{13}$.

20. Find the seventh and eleventh terms of $(x^2 - y\sqrt[3]{x})^{14}$.

21. Find the sixth and ninth terms of $(\frac{1}{2}a^2b - 2\sqrt[3]{a})^{12}$.

Find the ratio of

22. The third to the fifth term in the expansion of

$$\left(1 + \frac{x\sqrt{y}}{3}\right)^{12}$$

23. The tenth and twelfth terms of $\left(x^{\frac{2}{3}} + \frac{y}{2\sqrt{x}}\right)^{20}$.

24. Find the middle term of $(3a^{\frac{3}{4}} - x\sqrt[3]{a})^{10}$.

Write the formula for

25. The $r + 1$st term of $(x + a)^n$.

26. The $r - 1$st term. For the $r + 3$d term.

27. The rth term of $(x + a)^{n+2}$.

28. Term containing x^5 in $\left(x - \frac{2}{x}\right)^{11}$.

29. Term containing x^{18} in $\left(x^2 - \frac{a}{x}\right)^{15}$.

30. Term containing x^{24} in $\left(\frac{x}{2} + \sqrt{x^3}\right)^{18}$.

31. Term not containing x in $\left(x^3 - \frac{2}{x}\right)^{12}$.

32. Term containing x in $\left(y\sqrt{x} + \sqrt[3]{\frac{y}{x}}\right)^{17}$.

33. By use of the binomial formula find the value of $(1.1)^{12}$ to three decimal places.

SUG. Expand $(1 + .1)^{12}$.

Find the value of

34. $(1.2)^{12}$. **35.** $(1.3)^8$. **36.** $(2.2)^8$.

37. Find the coefficient of x^6 in $\left(x - \frac{2}{x}\right)^{14}$.

38. In the shortest way find the 98th term of $\left(2a - \frac{1}{\sqrt{a}}\right)^{100}$.

39. Expand $(x + a)^{n+3}$ to 4 terms.

40. Expand $(x + a)^{n-2}$ to 5 terms.

41. Expand $(1 - 1)^n$ by the binomial theorem.

42. Prove that in the binomial formula the sum of the coefficients of the odd terms equals the sum of the coefficients of the even terms.

43. Prove that the sum of the coefficients of the terms in the expansion of $(a + b)^{10}$ is 2^{10}. That the sum of the coefficients in the expansion of $(a + b)^n$ is 2^n.

44. Who discovered the binomial theorem and when? (See p. 464.) Find out all you can about this man.

45. State the advantages or utilities in the binomial theorem.

46. Make up and work three examples similar to such of the above as the teacher may indicate.

47. Practice oral work with exponents as in Exercise 93 (p. 303).

CHAPTER XXVI

HISTORY OF ELEMENTARY ALGEBRA

295. Epochs in the Development of Algebra. Some knowledge of the origin and development of the symbols and processes of algebra is important to a thorough understanding of the subject.

The oldest known mathematical writing is a papyrus roll, now in the British Museum, entitled "Directions for Attaining to the Knowledge of All Dark Things." It was written by a scribe named Ahmes at least as early as 1700 B. C., and is a copy, the writer says, of a more ancient work, dating, say, 3000 B. C., or several centuries before the time of Moses. This papyrus roll contains, among other things, the beginnings of algebra as a science. Taking the epoch indicated by this work as the first, the principal epochs in the development of algebra are as follows:

1. **Egyptian: 3000 B. C.–1500 B. C.**

2. **Greek** (at Alexandria): **200 A. D.–400 A. D.** Principal writer, Diophantus.

3. **Hindoo** (in India): **500 A. D.–1200 A. D.**

4. **Arab: 800 A. D.–1200 A. D.**

5. **European: 1200 A. D.–.** Leonardo of Pisa, an Italian, published in 1202 A. D. a work on the Arabic arithmetic which contained also an account of the science of algebra as it then existed among the Arabs. From Italy the knowledge

of algebra spread to France, Germany, and England, where its subsequent development took place.

We will consider briefly the history of

I. Algebraic Symbols.
II. Ideas of Algebraic Quantity.
III. Algebraic Processes.

I. History of Algebraic Symbols

296. Symbol for the Unknown Quantity.

1. **Egyptians** (1700 B. C.): used the word *hau* (expressed, of course, in hieroglyphics), meaning " heap."

2. **Diophantus** (Alexandria, 350 A. D.?): ς', or $\varsigma^{o'}$; plural, $\varsigma\varsigma$.

3. **Hindoos** (500 A. D.–1200 A. D.): Sanscrit word for " color," or first letters of words for colors (as blue, yellow, white, etc.).

4. **Arabs** (800 A. D.–1200 A. D.): Arabic word for " thing " or " root " (the term *root*, as still used in algebra, originates here).

5. **Italians** (1500 A. D.): *Radix, R, Rj.*

6. **Bombelli** (Italy, 1572 A. D.): ①

7. **Stifel** (Germany, 1544): *A, B, C, ...*

8. **Stevinus** (Holland, 1586): ①

9. **Vieta** (France, 1591): vowels *A, E, I, O, U.*

10. **Descartes** (France, 1637): x, y, z, etc.

297. Symbols for Powers (of x at first); Exponents.

1. Diophantus: δυναμις, or $\delta^{\bar{v}}$ (for square of the unknown quantity); κυβος, or $\kappa^{\bar{v}}$ (for its cube).

2. Hindoos: initial letters of Sanscrit words for " square " and " cube."

3. Italians (1500 A. D.): "census" or "zensus" or "*z*" (for x^2); "cubus" or "*c*" (for x^3).
4. Bombelli (1579): $\overset{1}{\smile}$, $\overset{2}{\smile}$, $\overset{3}{\smile}$, (for x, x^2, x^3).
5. Stevinus (1586): ①, ②, ③, (for x, x^2, x^3).
6. Vieta (1591): *A*, *A quadratus*, *A cubus* (for x, x^2 x^3).
7. Harriot (England, 1631): *a*, *aa*, *aaa*.
8. Herigone (France, 1634): *a*, *a*2, *a*3.
9. Descartes (France, 1637): x, x^2, x^3.

Wallis (England, 1659) first justified the use of fractional and negative exponents, though the use of fractional exponents had been suggested earlier by Oresme (1350), and the use of negative exponents by Choquet (c. 1500).

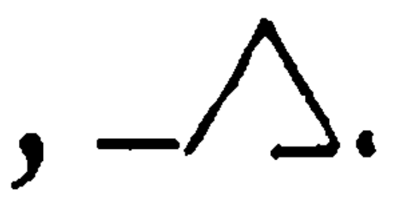
Newton (England, 1676) first used a general exponent, as in x^n, where n denotes any exponent, integral or fractional, positive or negative.

298. Symbols for Known Quantities.

1. Diophantus: μοναδες (i. e. monads), or $\mu^{\hat{o}}$.
2. Regiomontanus (Germany, 1430): letters of the alphabet.
3. Italians: *d*, from *dragma*.
4. Bombelli: $\overset{0}{\smile}$.
5. Stevinus: ⓪.
6. Vieta: consonants, *B*, *C*, *D*, *F*, ...
7. Descartes: *a*, *b*, *c*, *d*.

Descartes possibly used the last letters of the alphabet, x, y, z, to denote unknown quantities because these letters are less used and less familiar than a, b, c, d, ... , which he accordingly used to denote known numbers.

299. Addition Sign. The following symbols were used:

1. Egyptians: pair of legs walking forward (to the left), .
2. Diophantus: juxtaposition (thus, ab, meant $a + b$).

3. Hindoos: juxtaposition (survives in Arabic arithmetic, as in $2\frac{3}{5}$, which means $2 + \frac{3}{5}$).
4. Italians: *plus*, then *p* (or *e*, or *ϕ*).
5. Germans (1489): -+, +, +.

300. Subtraction Sign.

1. Egyptians: pair of legs walking backward (to the right), thus, △_; or a flight of arrows.
2. Diophantus: ᛦ (Greek letter ψ inverted).
3. Hindoos: a dot over the subtracted quantity (thus, $m\dot{n}$ meant $m - n$).
4. Italians: *minus*, then *M* or *m* or *de*.
5. Germans (1489): horizontal dash, –.

The signs + and – were first printed in Johann Widman's *Mercantile Arithmetic* (1489). These signs probably originated in German warehouses, where they were used to indicate excess or deficiency in the weight of bales and chests of goods. Stifel (1544) was the first to use them systematically to indicate the operations of addition and subtraction.

301. Multiplication Sign. Multiplication at first was usually expressed in general language. But

1. Hindoos indicated multiplication by the syllable *bha*, from *bharita*, meaning "product," written after the factors.
2. Oughtred and Harriot (England, 1631) invented the present symbol, ×.
3. Descartes (1637) used a dot between the factors (thus, $a \cdot b$).

302. Division Sign.

1. Hindoos indicated division by placing the divisor under the dividend (no line between). Thus, $\substack{c \\ d}$ meant $c \div d$.
2. Arabs, by a straight line. Thus, $a - b$, or $a \mid b$, or $\frac{a}{b}$.

3. Italians expressed the operation in general language.
4. Oughtred, by a dot between the dividend and divisor.
5. Pell (England, 1630), by ÷.

303. Equality Sign.

1. Egyptians: ∠ (Also other more complicated symbols to indicate different kinds of equality).

2. Diophantus: general language or the symbol, ι.

3. Hindoos: by placing one side of an equation immediately under the other side.

4. Italians: *æ* or *a*; that is, the initial letters of *æqualis* (equal). This symbol was afterward modified into the form, ∞, and was much used, even by Descartes, long after the invention of the present symbol by Recorde.

5. Recorde (England, 1540): =.

He says that he selected this symbol to denote equality because "than two equal straight lines no two things can be more equal."

304. Other Symbols used in Elementary Algebra.

Inequality Signs, $>$ $<$, were invented by Harriot (1631).

Oughtred, at the same time, proposed ⊐, ⊐ as signs of inequality, but those suggested by Harriot were manifestly superior.

Parenthesis, ( ), was invented by Girard (1629).

The **Vinculum** had been previously suggested by Vieta (1591).

Radical Sign. The Hindoos used the initial syllable of the word for square root, *Ka,* from *Karania,* to indicate square root.

Rudolph (Germany, 1525) suggested the symbol used at present, √, (the initial letter, *r*, in the script form, of the word *radix,* or root) to indicate square root, ʌw to denote the 4th root, and ʌww to denote cube root.

Girard (1633) denoted the 2d, 3d, 4th, etc., roots, as at present, by $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, etc.

The sign for **Infinity,** ∞, was invented by Wallis (1649).

305. Other Algebraic Symbols have been invented in recent times, but these do not belong to elementary algebra.

Other kinds of algebra have also been invented, employing other systems of the symbols.

306. General Illustration of the Evolution of Algebraic Symbols. The following illustration will serve to show the principal steps in the evolution of the symbols of algebra:

At the time of Diophantus the numbers 1, 2, 3, 4, ... were denoted by letters of the Greek alphabet, with a dash over the letters used; as, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, ...

In the algebra of Diophantus the coefficient occupies the last place in a term instead of the first as at present.

Beginning with Diophantus, the algebraic expression, $x^2 + 5x - 4$, would be expressed in symbols as follows:

$\delta^{\upsilon}\bar{\alpha}\ \varsigma\acute{o}\ \bar{\epsilon}\ \pitchfork\ \mu^{\circ}\ \bar{\delta}$	(Diophantus, 350 A. D.)
$1z\ p.5\ Rm.4$	(Italy, 1500 A. D.).
$1Q + 5N - 4$	(Germany, 1575).
$1^2\,p.5^1\,m.4\textcircled{0}$	(Bombelli, 1579).
$1\textcircled{2} + 5\textcircled{1} - 4\textcircled{0}$	(Stevinus, 1586).
$1Aq + 5A - 4\textcircled{0}$	(Vieta, 1591).
$1aa + 5a - 4$	(Harriot, 1631).
$1a2 + 5a1 - 4$	(Herigone, 1634).
$x^2 + 5x - 4$	(Descartes, 1637).

307. Three Stages in the Development of Algebraic Symbols.

1. **Algebra without Symbols** (called **Rhetorical Algebra**). In this primitive stage, algebraic quantities and operations were expressed altogether in words, without the use of sym-

bols. The Egyptian algebra and the earliest Hindoo, Arabian, and Italian algebras were of this sort.

2. **Algebra in which the Symbols are Abbreviated Words** (called **Syncopated Algebra**). For instance, *p* is used for *plus*. The algebra of Diophantus was mainly of this sort. European algebra did not get beyond this stage till about 1600 A. D.

3. **Symbolic Algebra.** In its final or completed state, algebra has a system of notation or symbols of its own, independent of ordinary language. Its operations are performed according to certain laws or rules, "independent of, and distinct from, the laws of grammatical construction."

Thus, to express addition in the three stages we have *plus*, *p*, $+$; to express subtraction, *minus*, *m*, $-$; to express equality, *æqualis*, *æ*, $=$.

Along with the development of algebraic symbolism, there was a corresponding development of ideas of algebraic quantity and of algebraic processes.

II. HISTORY OF ALGEBRAIC QUANTITY

308. The Kinds of Quantity considered in algebra are positive and negative; particular (or numerical) and general; integral and fractional; rational and irrational; commensurable and incommensurable; constant and variable; real and imaginary.

309. Ahmes (1700 B. C.) in his treatise uses *particular, positive* quantity, both *integral* and *fractional* (his fractions, however, are usually limited to those which have a unity for a numerator). That is, his algebra treats of quantities like 8 and $\frac{1}{6}$, but not like -3, or $-\frac{2}{5}$, or $\sqrt{2}$, or $-a$.

310. Diophantus (350 A. D.) used *negative* quantity, but only in a limited way; that is, in connection with a larger positive quantity. Thus, he used 7 − 5, but not 5 − 7, or − 2. He did not use, nor apparently conceive of, negative quantity having an independent existence.

311. The **Hindoos** (500 A. D.–1200 A. D.) had a distinct idea of independent or *absolute negative* quantity, and used the minus sign both as a quality sign and a sign of operation. They explained independent negative quantity much as it is explained to-day by the illustration of debts as compared with assets, and by the opposition in direction of two lines.

Pythagoras (Greece, 520 B. C.) discovered *irrational* quantity, but the Hindoos were the first to use this in algebra.

312. The **Arabs** avoided the use of negative quantity as far as possible. This led them to make much use of the process of transposition in order to get rid of negative terms in an equation. Their name for algebra was "al gebr we'l mukabala," which means "transposition and reduction."

The Arabs used *surd* quantities freely.

313. In **Europe** the free use of absolute negative quantity was restored.

Vieta (1591) was principally instrumental in bringing into use *general algebraic quantity* (known quantities denoted by letters and not figures).

Cardan (Italy, 1545) first discussed *imaginary* quantities, which he termed "sophistic" quantities.

Euler (Germany, 1707–83) and Gauss (Germany, 1777–1855) first put the use of *imaginary* quantities on a scientific basis. The symbol i for $\sqrt{-1}$ was suggested by Gauss.

Descartes (1637) introduced the systematic use of *variable* quantity as distinguished from constant quantity.

III. History of Algebraic Processes

314. Solution of Equations. **Ahmes** solved many *simple equations of the first degree,* of which the following is an example:

"Heap its seventh, its whole equals nineteen. Find heap."

In modern symbols this is,

$$\text{Given } \frac{x}{7} + x = 19; \text{ find } x.$$

The correct answer, $16\frac{5}{8}$, was given by Ahmes.

Hero (Alexandria, 120 B. C.) solved what is in effect the *quadratic equation,*

$$\tfrac{11}{14}d^2 + \tfrac{29}{7}d = s$$

where d is unknown, and s is known.

Diophantus solved simple equations of one unknown quantity, and *simultaneous equations* of two and three unknown quantities. He solved quadratic equations much as is done at present, completing the square by the method given in Art. 226. However, in order to avoid the use of negative quantity as far as possible, he made three classes of quadratic equations, thus,

$$\left\{\begin{array}{l} ax^2 + bx = c, \\ ax^2 + c = bx, \\ ax^2 = bx + c. \end{array}\right.$$

In solving quadratic equations, he rejected negative and irrational answers.

He also solved equations of the form $ax^m = bx^n$.

He was the first to investigate *indeterminate equations,* and solved many such equations of the first degree with two or three unknown quantities, and some of the second degree.

The **Hindoos** first invented a *general method of solving a quadratic equation* (now known as the Hindoo method, see Art. 233). They also solved particular cases of higher degrees, and gave a general method of solving indeterminate equations of the first degree.

The **Arabs** took a step backward, for, in order to avoid the use of negative terms; they made six cases of quadratic equations; viz.:

$$\begin{array}{ll} ax^2 = bx, & ax^2 + bx = c, \\ ax^2 = c, & ax^2 + c = bx, \\ bx = c, & ax^2 = bx + c. \end{array}$$

Accordingly, they had no general method of solving a quadratic equation.

The Arabs, however, solved equations of the form $ax^{2p}+bx^p = c$, and obtained a geometrical solution of *cubic equations* of the form $x^3 + px + q = 0$.

In **Italy**, Tartaglia (1500–1559) discovered the *general solution of the cubic equation*, now known as Cardan's solution. Ferrari, a pupil of Cardan, discovered the solution of *equations of the fourth degree.*

Vieta discovered many of the elementary *properties of an equation of any degree;* as, for instance, that the number of the roots of an equation equals the degree of the equation.

315. Other Processes. Methods for the **addition, subtraction,** and **multiplication** of polynomial expressions were given by Diophantus.

Transposition was first used by Diophantus, though, as a process, it was first brought into prominence by the Arabs. The word *algebra* is an Arabic word and means " transposition " (*al* meaning " the," and *gebr* meaning " transposition").

The Greeks and Romans had a very limited knowledge of

fractions. The Hindoos seem to have been the first to reduce fractions to a common denominator.

The **square** and **cube root** of polynomial expressions were extracted by the Hindoos.

The methods for using **radicals**, including the extraction of the square root of binomial surds and the rationalizing of the denominators of fractions, were also invented by the Hindoos.

The methods of using fractional and negative **exponents** were determined by Wallis (1659) and Sir Isaac Newton.

The three **progressions** were first used by Pythagoras (569 B. C.–500 B. C.)

Permutations and **combinations** were investigated by Pascal and Fermat (France, 1654).

The **binomial theorem** was discovered by Newton (1655), and, as one of the most notable of his many discoveries, is said to have been engraved on his monument in Westminster Abbey.

Graphs of the kind treated in this book were first invented by Descartes (France, 1637).

The fundamental **laws of algebra** (the Associative, Commutative, and Distributive Laws; see Arts. 316–317) were first clearly formulated by Peacock and Gregory (England, 1830–45), though, of course, the existence of these laws had been implicitly assumed from the beginnings of the science.

Students who desire to investigate the history of algebra in more detail should read the second part of Fine's *Number System of Algebra*, Ball's *Short History of Mathematics*, and Cajori's *History of Elementary Mathematics*.

APPENDIX

FUNDAMENTAL LAWS OF ALGEBRA

316. The following **Laws of Algebra** have been used in the preceding pages without formal statement:

A. **The Commutative Law** (or Law of Order).

1. *For addition,* $a + b = b + a$.
2. *For multiplication,* $ab = ba$.
3. *For division,* $a \div b \times c = a \times c \div b$.

B. **The Associative Law** (or First Law of Grouping).

1. *For addition,* $a + b + c = a + (b + c) = (a + b) + c$.
2. *For multiplication,* $abc = a(bc) = (ab)c$.

C. **The Distributive Law** (or Second Law of Grouping).

1. *For multiplication,* $a(b + c) = ab + ac$. Hence, inversely, $ab + ac = a(b + c)$.
2. *For division,* $\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$.

Who first formulated the laws of algebra? (See p. 464.)

317. Utility of the Laws of Algebra. The laws stated in Art. 316 are methods adopted for arranging and grouping algebraic symbols so as to decrease the amount of work and to increase the importance of the results attained.

Thus, in the following example we are able to eliminate the parenthesis by use of the Distributive Law and to collect terms by use of the Commutative and Associative Laws.

Ex. $6(x+y)+3(x-y+z)+2(x+2y-z)$.

$$\begin{aligned} &= 6x+6y+3x-3y+3z+2x+4y-2z \\ &= 6x+3x+2x+6y-3y+4y+3z-2z \\ &= 11x+7y+z \textit{ Ans.} \end{aligned}$$

The use of these laws enables us to diminish the 23 symbols used in the first expression to the 8 symbols used in the last expression.

It should be noted that by changing the laws stated in Art. 316, kinds of algebra different from that presented in this book, and adapted to other uses, may be devised. Thus, in a certain important kind of algebra $ab = -ba$, not ba.

Even in arithmetic the commutative law holds only in a limited way. For, while $5 \times 7 = 7 \times 5$, 57 does not equal 75.

Detached Coefficients

318. Examples.

Ex. 1. Multiply $x^3+3a^2x-2a^3$ by $x^3-4ax^2+3a^3$.

$$\begin{array}{rrrrrrr}
1 & +0 & +3 & -2 & & & \\
1 & -4 & +0 & +3 & & & \\
\hline
1 & +0 & +3 & -2 & & & \\
 & -4 & -0 & -12 & +8 & & \\
 & & + & 3 & +0 & +9 & -6 \\
\hline
1 & -4 & +3 & -11 & +8 & +9 & -6
\end{array}$$

Hence, $x^6-4ax^5+3a^2x^4-11a^3x^3+8a^4x^2+9a^5x-6a^6$ *Product*

Let the pupil work this example in full and compare the labor in the two processes.

Ex. 2. Divide $x^4-2x^2y^2+8xy^3-3y^4$ by $x^2+2xy-y^2$.

$$\begin{array}{rrrrr|l}
1 & +0 & -2 & +8 & -3 & 1+2-1 \\
\cline{6-6}
1 & +2 & -1 & & & 1-2+3 \\
\cline{1-5}
 & -2 & -1 & +8 & & \\
 & -2 & -4 & +2 & & \\
\cline{2-5}
 & & 3 & +6 & -3 & \\
 & & 3 & +6 & -3 & \\
\cline{3-5}
\end{array}$$

Hence, $x^2-2xy+3y^2$ *Quotient*

EXERCISE 151

By use of detached coefficients, work such examples in Exercises 17 (p. 68) and 21 (p. 85) as the teacher may indicate.

FACTOR THEOREM

319. Illustrations. The method by which an expression is prepared for division at sight and hence for factoring, as explained in Ex. 3, of Art. 99 (p. 150), may be carried further and then abbreviated.

Ex. 1. Factor $x^4 - 4x^3 - x^2 + 16x - 12$.

We may test this expression as to its divisibility by $x - 1$ by splitting terms in succession thus,

$$\begin{aligned} &x^4 - 4x^3 - x^2 + 16x - 12 \\ &\qquad = x^4 - x^3 - 3x^3 - x^2 + 16x - 12 \\ &\qquad = x^4 - x^3 - 3x^3 + 3x^2 - 4x^2 + 16x - 12 \\ &\qquad = x^4 - x^3 - 3x^3 + 3x^2 - 4x^2 + 4x + 12x - 12 \\ &\qquad = x^3(x-1) - 3x^2(x-1) - 4x(x-1) + 12(x-1) \end{aligned}$$

Hence, $x - 1$ is a factor of the original expression.

This result might have been obtained in a shorter way by observing that, as this last expression reduces to zero when $x = 1$, the first expression, might be tested as to its divisibility by $x - 1$ by substituting 1 for x and noting whether the expression reduces to zero. This last test may be further abbreviated to a matter of noting whether the algebraic sum of the coefficients of the terms is zero.

Ex. Determine by inspection whether $x^4 + x^3 - 6x^2 - 4x + 8$ is divisible by $x - 1$.

Summing the coefficients, we have

$$1 + 1 - 6 - 4 + 8 = 0;$$

hence, $x - 1$ is a factor of the given expression.

In like manner, if an expression is divisible by $x + 1$, the sum of the coefficients of the even terms must equal the sum of the coefficients of the odd terms.

320. Factor Theorem. *If any rational integral expression containing x becomes equal to zero, when a is substituted for x, then $x-a$ is a factor of the given expression.*

For, let E stand for any rational integral algebraic expression. If E is divided by $x - a$ till a remainder is obtained in which x does not occur, denote the quotient by Q and the remainder by R. Then

$$E \equiv Q(x-a)+R$$

Let $x = a$, then

$$0 = Q(0) + R \text{ (since } E = 0 \text{ when } x = a\text{)}$$

$$\therefore R = 0.$$

Hence, $E = Q(x-a)$, or $x - a$ is a factor of E

Ex. Factor $x^3 - 12x + 16$.

By trial we find that $x^3 - 12x + 16 = 0$

When $x = 2$

$$\therefore x - 2 \text{ is a factor of } x^3 - 12x + 16.$$

By division $x^3 - 12x + 16 = (x-2)(x^2+2x-8)$

$= (x-2)(x-2)(x+4)$ *Factors*

Note that the only numbers which need be tried as values of x are the factors of the last term of the given expression. This follows from the fact that the last term of the dividend must be divisible by the last term of the divisor.

EXERCISE 152

Factor by use of the factor theorem:

1. $x^2 - 4$.
2. $x^2 - 3x - 28$.
3. $a^3 - b^3 + 3(a-b)$.
4. $(a-b)^2 + 3(a-b)$.
5. $a^3 + 5a - 6$.
6. $2x^2 + 7x - 15$.
7. $2x^3 - x^2 - 7x + 6$.
8. $4x^3 - 4x^2 - 14x - 6$.
9. $3x^3 + 8x^2 + 3x - 2$.
10. $2x^4 + x^3 - 14x^2 + 5x + 6$.
11. $6x^4 - 13x^3 - 45x^2 - 2x + 24$.
12. $x^3 - 2x^2 + 1$.
13. $x^3 - 6x^2 + 25$.
14. $x^4 - 28x^2 + 33x - 90$.

15. Prove that $x^n - y^n$ is always divisible by $x - y$.

16. Prove that $x^n + y^n$ is divisible by $x + y$ when n is odd.

17. Show that $(1 - x)^2$ is a factor of $1 - x - x^n + x^{n+1}$.

18. Make up and work an example similar to Ex. 5. To Ex. 7.

H. C. F. AND L. C. M. OBTAINED BY LONG DIVISION

321. H. C. F. by Long Division. For polynomials that cannot be readily factored, the H. C. F. is found by the same general method that is used in arithmetic to determine the G. C. D. of large numbers.

Ex. Find the G. C. D. of 65 and 117.

$$\begin{array}{l}
65)117(1 \\
\quad\;\;\underline{65} \\
\quad\;\;52)65(1 \\
\qquad\quad\;\underline{52} \\
\qquad\quad\;13)52(4 \\
\qquad\qquad\;\;\underline{52}
\end{array}
\qquad \therefore \text{ G. C. D. of 65 and 117 is 13}$$

In applying the above method to algebraic expressions, note, for instance, that the H. C. F. of $x^2 - 4$ and $x^2 - 3x + 2$ is the same as the H. C. F. of $5x(x^2 - 4)$ and $2a(x^2 - 3x + 2)$; the H. C. F. in either case is $x - 2$.

Or, in general, if m, n, P, and Q are algebraic expressions, the H. C. F. of P and Q is the same as the H. C. F. of mP and nQ, provided that m has no factor which is a factor of Q and n has no factor which is a factor of P.

This property of algebraic expressions enables us to simplify the process of finding the H. C. F. by multiplying or dividing one of the algebraic expressions by an expression which is not a factor of the other expression.

Ex. 1. Find the H. C. F. of $4x^3 - 4x^2 - 5x + 3$ and $10x^2 - 19x + 6$.

To render the first expression divisible by the second, we may multiply the first expression by 5, which is not a factor of the second expression.

The work may then be conveniently arranged as follows:

$$\begin{array}{r|r|l}
10x^2 - 19x + 6 & 4x^3 - 4x^2 - 5x + 3 & \\
 & 5 & \\
\cline{2-2}
 & 20x^3 - 20x^2 - 25x + 15 & 2x \\
 & 20x^3 - 38x^2 + 12x & \\
\cline{2-2}
 & 18x^2 - 37x + 15 & \\
 & 5 & \\
\cline{2-2}
 & 90x^2 - 185x + 75 & 9 \\
10x^2 - 15x & 90x^2 - 171x + 54 & \\
\cline{1-2}
-4x + 6 & -7\,\big|\boxed{-14x + 21} & \\
-4x + 6 & \text{H. C. F} = 2x - 3 & 5x - 2 \\
\cline{1-1}
\end{array}$$

The work may be shortened by the use of detached coefficients (see Art. 318).

Ex. 2. Find the H. C. F. of $8x^4 - 8x^3 - 10x^2 + 6x$ and $30x^4 - 57x^3 + 18x^2$.

$$8x^4 - 8x^3 - 10x^2 + 6x = 2x(4x^3 - 4x^2 - 5x + 3)$$
$$30x^4 - 57x^3 + 18x^2 = 3x^2(10x^2 - 19x + 6)$$

The H. C. F. of $2x$ and $3x^2$ is x.

The H. F. of $4x^3 - 4x^2 - 5x + 3$ and $10x^2 - 19x + 6$, by the method of Ex. 1, is found to be $2x - 3$.

Hence, the complete H. C. F. is $x(2x - 3)$ *Ans.*

322. L. C. M. by Long Division.

Ex. Find the L. C. M. of 182 and 299.

By the division method, the G. C. D. is found to be 13.

Then, since $182 = 13 \times 14, \qquad 299 = 13 \times 23,$

$$\begin{array}{r|l}
13 & 13 \times 14 \quad , \quad 13 \times 23 \\
\hline
 & \phantom{13 \times{}} 14 \quad , \quad \phantom{13 \times{}} 23
\end{array}$$

$$\therefore\ \text{L. C. M} = 13 \times 14 \times 23.$$

Similarly, to find the L. C. M. of two algebraic expressions which cannot be readily factored, we first find the H. C. F. of the two expressions by the division method.

Ex. Find the L. C. M. of $4x^2 + 3x - 10$ and $4x^3 + 7x^2 - 3x - 15$.

We first find the H. C. F. by the division method; this is $4x - 5$.

Then
$$4x^2 + 3x - 10 = (4x - 5)(x + 2)$$
$$4x^3 + 7x^2 - 3x - 15 = (4x - 5)(x^2 + 3x + 3)$$
$$\therefore \text{L. C. M.} = (4x - 5)(x + 2)(x^2 + 3x + 3) \quad \textit{Ans.}$$

EXERCISE 153

Find the H. C. F. and L. C. M. of

1. $2x^2 - x - 3$ and $4x^3 - 4x^2 - 3x + 5$.

2. $6x^2 - x - 12$ and $6x^3 - 13x^2 - 6x + 18$.

3. $x^3 + x^2 + x - 3,\ x^3 - 3x^2 + 5x - 3$.

4. $3x^3 - 9x^2 + 9x - 3,\ 6x^3 - 6x^2 - 6x + 6$.

5. $6x^4 - 5x^3 + 6x^2 + 5x,\ 2x^4 - 9x^3 - 9x^2 - 2x$.

6. $9x^5 + 3x^4 - 3x^3 + 12x^2,\ 18x^4 + 42x^3 + 6x^2 - 24x$.

7. $3x^3 + 7x^2 - 5x + 3,\ 2x^3 + 3x^2 - 7x + 6$.

8. $x^4 - x^3 - x^2 + 7x - 6,\ x^4 + x^3 - 5x^2 + 13x - 6$.

9 $2x^3 - 16x + 6,\ 5x^6 + 15x^5 + 5x + 15$.

10. $2x^5 + x^4 + 2x^3 - x^2 - 1,\ 5x^4 + 2x^3 + 3x^2 - 2x + 1$.

11. $3x^4 + 2x^3y + 2x^2y^2 + 5xy^3 - 2y^4,\ 6x^4 + x^3y + 2x^2y^2 + 2xy^3 - y^4$.

12. $3x^5 + 2x^4 - 8x^3 - 3x^2 + 4x,\ 3x^5 - 10x^4 + 14x^3 - 11x^2 + 4x$.

The H. C. F. (or L. C. M.) of three or more expressions may be obtained by finding that of two of them; then find the H. C. F. (or L. C. M.) of this result and another of the quantities; the last H. C. F. (or L. C. M.) thus obtained is the one required.

13. $x^3 - x^2 - x - 2,\ x^3 - 2x^2 + 3x - 6,\ 2x^3 - 3x^2 - x - 2$.

14. $2x^4 - 14x^2 + 12x,\ 2x^4 + 6x^3 - 32x^2 + 24x,\ 6x^4 - 30x^3 + 42x^2 - 18x$.

CUBE AND HIGHER ROOTS

323. Cube Root of Polynomials. A general method for determining the cube root of any polynomial which is a perfect cube may be found by studying the relation between the terms of a binomial — or, in general, of a polynomial — and the terms of its cube (as $a + b$, and its cube, $a^3 + 3a^2b + 3ab^2 + b^3$). This relation stated in the inverse form gives the method for extracting the cube root.

The essence of this method consists in writing $a^3 + 3a^2b + 3ab^2 + b^3$ in the form $a^3 + b(3a^2 + 3ab + b^2)$.

Ex. Extract the cube root of $x^6 + 3x^5 - 5x^3 + 3x - 1$.

$$\begin{array}{rcr|l}
 & & & \underline{x^2 + x - 1}\ \textit{Root} \\
 & & & x^6 + 3x^5 - 5x^3 + 3x - 1 \\
 & & & \underline{x^6 \qquad\qquad\qquad\qquad\qquad} \\
3(x^2)^2 & = & 3x^4 \qquad\qquad & 3x^5 - 5x^3 \\
3(x^2)x + x^2 & = & \underline{+3x^3 + x^2} & \\
\text{Complete divisor} & = & 3x^4 + 3x^3 + x^2 & \underline{3x^5 + 3x^4 + x^3 \qquad\qquad} \\
3(x^2 + x)^2 & = & 3x^4 + 6x^3 + 3x^2 & -3x^4 - 6x^3 + 3x - 1 \\
3(x^2 + x)(-1) + (-1)^2 & = & \underline{-3x^2 - 3x + 1} & \\
\text{Complete Divisor} & = & \underline{3x^4 + 6x^3 - 3x + 1} & \underline{-3x^4 - 6x^3 + 3x - 1}
\end{array}$$

Let the pupil state this process as a formal rule.

EXERCISE 154

Find the cube root of

1. $a^3 + 6a^2x + 12ax^2 + 8x^3$.
2. $27 - 27a + 9a^2 - a^3$.
3. $a^6 - 3a^5 - 3a^4 + 11a^3 + 6a^2 - 12a - 8$.
4. $12x^4 - 36x + 64x^6 - 6x^2 - 8 + 117x^3 - 144x^5$.
5. $95a^3 + 72a^4 - 72a^2 + 15a^5 + 15a + a^6 - 1$.
6. $114x^4 - 171x^2 - 27 - 135x + 8x^6 - 60x^5 + 55x^3$.
7. $x^3 - 3x^2 + 6x - 7 + \frac{6}{x} - \frac{3}{x^2} + \frac{1}{x^3}$.
8. $x^6 + \frac{6x^5}{y} + \frac{15x^4}{2y^2} - \frac{45x^2}{4y^4} + \frac{27x}{2y^5} - \frac{27}{8y^6} - \frac{10x^3}{y^3}$.

324. Cube Root of Arithmetical Numbers. The same general method as that used in Art. 323 can be used to extract the cube root of arithmetical numbers. The process is slightly different from the algebraic process, owing to the fact that all the numbers which compose a given cube are united or fused into a single number.

Thus, $$\begin{aligned}(42^3 = (40+2)^3 &= 40^3 + 3\times 40^2 \times 2 + 3 \times 40 \times 2^2 + 2^3\\ &= 64000 + 9600 + 480 + 8\\ &= 74088\end{aligned}$$

Reversing this process, we obtain a method of extracting the cube root of a number.

Ex. 1. Extract the cube root of 74088.

$$\begin{array}{lrr|l}
 & & & \overset{\frown}{74}\overset{\frown}{088}\,\underline{|42}\ \textit{Root}\\
 & 40^3 = & & \underline{64}\\
\text{Trial Divisor,} & 3\times 40^2 = & 4800 & 10088\\
 & 3\times 40\times 2 = & 240 & \\
 & 2^2 = & \underline{4} & \\
\text{Complete Divisor,} & = & 5044 & \underline{10088}
\end{array}$$

Ex. 2. Extract the cube root of $\frac{5}{12}$ to 4 decimal places.

$$\tfrac{5}{12} = .416666666666+$$

$$\begin{array}{rr|l}
 & & .\overset{\frown}{416}\overset{\frown}{666}\overset{\frown}{666}+\,\underline{|.7469+}\ \textit{Root}\\
 & & \underline{343}\\
3\times(70)^2 = & 14700 & 73666\\
3\times(70\times 4) = & 840 & \\
4^2 = & \underline{16} & \\
 & 15556 & \underline{62224}\\
3\times(740)^2 = & 1642800 & 11442666\\
3\times(740\times 6) = & 13320 & \\
6^2 = & \underline{36} & \\
 & 1656156 & \underline{9936936}\\
3\times(7460)^2 = & 166954800 & 1505730666\\
 & & \underline{1502593200}\\
 & & 3137466
\end{array}$$

The first three figures of the root are found directly. The last figure is then found by division of the remainder, using three times the square of the root already found as a divisor. The number of figures of the root that may thus be found by division is two less than the number of figures already found.

Let the pupil state the above process as a rule.

EXERCISE 155

Find the cube root of

1. 3375. **4.** 43614208. **7.** 344324.701729.

2. 753571. **5.** 32891033664. **8.** .000127263527.

3. 1906624. **6.** 520688691.125. **9.** 0.991026973..

Find to three decimal places the cube root of

10 75. **12.** 5.6. **14.** $7\frac{5}{11}$. **16.** $\frac{1}{16}$. **18.** $1\frac{1}{15}$.

11. 6. **13.** $3\frac{2}{3}$. **15.** $19\frac{7}{9}$. **17.** $\frac{1}{101}$. **19.** $8\frac{1}{25}$.

Compute the value of

20. $\sqrt{5+2\sqrt[3]{5}}$. **21.** $\sqrt[3]{3\sqrt{10}-2\sqrt{10}}$.

22. $\sqrt{3\sqrt{0.8}-2\sqrt{1.935}}$.

Visualize the following objects by the aid of cube root:

23. 150,000,000 cu. yd. of earth.

24. 40,000,000,000 feet of lumber.

25. 50,000,000 tons of iron (taking 480 lb. as the weight of one cubic foot of iron).

26. Make up and work an example similar to Ex. 12. To Ex. 23. Ex. 25.

325. Higher Roots Obtained by Successive Extractions. By the law of exponents, the square of the square of any quantity gives the fourth power of the quantity. Hence, reversing the process, the *fourth root* of a quantity is the *square* *root* of the *square root* of the quantity. Similarly, the *sixth root* of a quantity is the *square root* of the *cube root* of the quantity. The *eighth, ninth, tenth* . . . . roots of a quantity may be found by similar methods.

Ex. Extract the fourth root of

$$81a^4 + 108a^3 + 54a^2 + 12a + 1$$

Obtain first the square root of the given expression, which is $9a^2 + 6a + 1$. Extracting the square root of this, we obtain $3a + 1$, the fourth root of the original expression.

EXERCISE 156

Find the fourth root of

1. 130321. **2.** 3418801. **3.** 90. **4.** .8.

5. $1 - 12ab + 54a^2b^2 - 108a^3b^3 + 81a^4b^4$.

6. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

7. $64x^2 - 56x^4 + 16x^5 + x^8 + 16 - 32x^3 + 16x^6 - 8x^7 + 64x$.

Find the sixth root of

8. 7529536. **9.** 1544804416. **10.** 15.

11. $x^6 + 1215x^2 + 729 - 1458x + 135x^4 - 540x^3 - 18x^5$.

12. $4096x^{12} - 3072x^{10} + 960x^8 - 160x^6 + 15x^4 - \frac{3}{4}x^2 + \frac{1}{64}$.

EXERCISE 157

Review of Algebra to Quadratics

1. If $x = 2$, $y = -3$, $z = -\frac{1}{2}$, find the value of

$$3xy - y(x + 4z) - xz(4y + 6x) + 3yz(x + y)(y + 2z).$$

2. Find the value of $\left(\dfrac{1}{1 + \sqrt{1 + x}} + \dfrac{1}{1 - \sqrt{1 + x}}\right)\sqrt{1 - x}$, when $x = \frac{4}{5}$.

3. Find the numerical value of the following expressions when $a = 3$ and $b = -3$: $3a^2 - 5b^2$; $(\sqrt{-b})^2$; $(\sqrt{b})^6$.

4. Simplify

$$(a+b)(a-b)-\{(a+b-c)-(b-a-c)+(b+c-a)\}(a-b-c).$$

5. Divide $x^4+x^3+ax^2+bx-3$ by x^2+2x-3, and find what values a and b must have if there is to be no remainder.

6. Factor:

(1) $x^{\frac{2}{3}}-9$.
(2) $x^{\frac{3}{5}}+27$.
(3) $4x-y^2$.
(4) $x^{2a}-y^{-6}$.
(5) $a^{\frac{3}{2}}-8b^{-1}$.
(6) $25n^5-y^{-2}$.

7. Factor:
(1) $xy-1-x+y$.
(2) $a^2+b^2+c^2-2ab+2ac-2bc$.
(3) $a^4x^4+a^2x^2+1$.
(4) $7(p-1)^2-27(p-1)+18$.

8. Factor:

(1) $3x-8x^{\frac{1}{2}}-35$.
(2) $10x^{\frac{1}{2}}-19x^{\frac{1}{4}}-56$.
(3) $12x^{\frac{4}{3}}+5x^{\frac{2}{3}}-72$.
(4) $a^{\frac{1}{3}}x^{\frac{1}{2}}-3a^{\frac{1}{3}}+5x^{\frac{1}{2}}-15$.
(5) $60-7\sqrt{3a}-6a$.
(6) $15x-2\sqrt{xy}-24y$.

9. Find the H. C. F. and the L. C. M. of

$$a^2+a^{\frac{3}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{3}{2}}-b^2 \text{ and } a^2-a^{\frac{3}{2}}b^{\frac{1}{2}}-a^{\frac{1}{2}}b^{\frac{3}{2}}-b^2.$$

10. The H. C F. of two expressions is $a(a-b)$, and their L. C. M. is $a^2b(a+b)(a-b)$. If one expression is $ab(a^2-b^2)$, find the other.

11. Find the L. C. M. of x^2+ax+a^2, x^3-a^3, and x^2-a^2.

12. Find the L. C. M. of $x^2+xy^{\frac{1}{2}}-2y$, $2x^2+5xy^{\frac{1}{2}}+2y$, and $2x^2-xy^{\frac{1}{2}}-y$.

13. Simplify

$$\frac{\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}}{\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}}-\frac{m}{n}-\frac{n}{m}.$$

Write the following expressions with positive exponents and simplify:

14. $(a-b)^{-1}+(a+b)^{-1}$.

15. $$\frac{(a+b)(a-b)^{-1}-(a-b)(a+b)^{-1}}{1-(a^2+b^2)(a+b)^{-2}}.$$

16. $a\,(1-a^{-2})\,(a+a^{-1})\,(1+a^{-1})^{-1}\,(a^2+1)^{-1}$.

17. Show that $\dfrac{n(n-1)}{2}+\dfrac{n(n-1)\,(n-2)}{6}=\dfrac{(n+1)\,n(n-1)}{6}$.

18. Simplify $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}}+\dfrac{1}{(1-x^2)^{\frac{1}{2}}}$.

19. Show that $\dfrac{a(b+c-a)}{(a-b)\,(c-a)}+\dfrac{b(c+a-b)}{(b-c)\,(a-b)}+\dfrac{c(a+b-c)}{(c-a)\,(b-c)}=2$.

20. What must be the value of n in order that $(2a+n)\div(3n+69a)$ may be equal to $\frac{1}{33}$, when $a=\frac{1}{3}$?

21. The equation $ax^2+bxy+cy^2=5$ is to be satisfied when $x=1$, $y=1$; when $x=2$, $y=-1$; and when $x=-1$, $y=2$. Find the values of a, b, and c.

Solve:

22. $\dfrac{3}{x}-5y=13.$
$\dfrac{5}{x}+\frac{1}{2}y=4.$

23. $4x-\dfrac{5}{3y}=-3.$
$3x+\dfrac{1}{4y}=-6\frac{1}{4}.$

24. $ax+by+cz=1.$
$bx+cy+az=1.$
$cx+ay+bz=1.$

25. $qx-rb=p(a-y).$
$\dfrac{qx}{a}+r=p\left(1+\dfrac{y}{b}\right).$

26. Given $A=2$, $B-2A=0$, $C-2B+3A=-3$, $D-2C+3B=-1$, and E $-\,2D+3C=0$; find the values of B, C, D, and E.

27. Extract the square root of $4x^n+9x^{-n}+28-24x^{-\frac{n}{2}}-16x^{\frac{n}{2}}$.

28. Obtain the square root of $a^4+b^4+2ab(a^2+b^2)+3a^2b^2$.

Simplify:

29. $8^{-\frac{2}{3}}+25^{\frac{3}{2}}-(\frac{1}{5})^{-3}+13^0-(\frac{1}{32})^{-\frac{3}{5}}$.

30. $(x^m)^{n-1}\cdot(x^n)^{m+1}\cdot(x^m)^{1-2n}$.

31. $\left[\left(x^{\frac{1}{a-c}}\right)^{a-\frac{c^2}{a}}\right]^{\frac{a}{a+c}}$.

32. $\dfrac{1-a^{\frac{1}{4}}b^{\frac{1}{3}}}{b^{\frac{1}{3}}\sqrt{a}-ab}\div\dfrac{(1+a^{\frac{1}{4}}b^{\frac{1}{3}})^{-1}}{a^{-\frac{1}{3}}b}$.

33. $\left(\dfrac{m^{q+p}}{m^q}\right)^p\div\left(\dfrac{m^q}{m^{q-p}}\right)^{p-q}$.

34. $\dfrac{a^{-2}b+5a^{-1}\sqrt{b}-66}{a^{-2}b+3a^{-1}\sqrt{b}-54}$.

35. $(.09)^{-\frac{3}{2}}$. Also $(-.064)^{\frac{2}{3}}$.

36. Find the ratio of $\frac{-6 \times 2^{-2}}{5}$ to $\frac{6^{-1} \times 2^{-2}}{5}$.

37. Simplify $\frac{x + y^{-1}}{x^{\frac{1}{3}} + y^{-\frac{1}{3}}} - \frac{x - y^{-1}}{x^{\frac{1}{3}} - y^{-\frac{1}{3}}}$.

38. Multiply $x^n + x^{\frac{n}{2}} + 1$ by $x^{-n} - x^{-\frac{n}{2}} + 1$.

39. If $p = \frac{1}{2}\left(\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} - \frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}\right)$, show that $(1 + p^2)^{\frac{1}{2}} = \frac{x + y}{2\sqrt{xy}}$.

40. Simplify $\sqrt{50} + \sqrt[6]{9} - 4\sqrt{\frac{1}{2}} + \sqrt{27} + \sqrt[9]{27} - \sqrt{64}$.

41. Find to 3 decimal places the value of $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2} - 1}$.

42. Simplify $\sqrt{(m - n)^2 a} + \sqrt{(m + n)^2 a} - \sqrt{am^2} + \sqrt{a(1 - m)^2} - \sqrt{a}$.

43. Which is the greatest, $\sqrt{\frac{4}{3}}$, $\sqrt{\frac{3}{2}}$, or $\sqrt[4]{\frac{7}{4}}$?

44. From $(c - x)\sqrt{c^2 - x^2}$ subtract $\sqrt{\frac{c + x}{c - x}}$.

45. Show that every even power of i is real (use i^{2n} as representing any even power of i). Also show that every odd power of i is imaginary.

46. Does $\frac{\sqrt{6}}{\sqrt{-2}}$ equal $\sqrt{-3}$?

47. Given $\sqrt{3} = 1.73205$. Compute the value of $\sqrt{\frac{256}{3}}$ in the shortest way.

48. Simplify $\frac{\sqrt{a^2 - x^2} + 2x^2(a^2 - x^2)^{-\frac{1}{2}}}{a^2 - x^2}$.

49. Find the value of $i^8 \div i^4$. Of i^{4n+1}. Of $1 \div i^3$.

50. A certain shelf will hold 20 geometries and 24 algebras, or 15 geometries and 36 algebras. How many geometries alone, or how many algebras alone, will the shelf hold?

51. A baseball nine has won .625 of the games it has played. If it has won 8 more games than it has lost, how many games has it played?

52. A certain solution is 45% alcohol. Water equal to what fractional part of the solution must be added to change it to a 25% solution?

53. A certain paint is half oil and half pigment. Oil equal to what fractional part of the given amount of paint must be added to make the oil equal to 60% of the whole?

54. Solve $\frac{9x+7}{2} - \left(x - \frac{x-2}{7}\right) = 36.$

55. Find the algebraic expression which, when divided by $x^2 - 2x + 1$, gives a quotient of $x^2 + 2x + 1$ and a remainder of $x - 1$.

56. Arrange $\sqrt{5}$, $\sqrt[4]{24}$, $\sqrt[3]{11}$ in descending order of magnitude.

57. Solve $mx + \frac{n}{y} = 1, \qquad nx + \frac{m}{y} = 1.$

58. Factor $x^{2n} + x^n y^{-n} + y^{-2n}$.

59. If a certain kind of cloth is 27 in. wide and loses 2% in width and 5% in length by shrinking, how many yards must a dressmaker buy in order that after shrinking it she shall have 20 sq. yd.?

60. Simplify $\frac{2x^2 - 3x + 1}{2x + \frac{4}{3x - 5}}(3x - 2)^2 \div (3x - 5).$

61. Express algebraically: 5 times the cube of a is divided by the fraction whose numerator is 6 times the square of b, and whose denominator is the square of the difference between x and twice the cube of y. Also express in words $\frac{5(a^3 + b^3)}{(x + 2y^4)^3}$.

62. The United States 5¢ piece (or nickel) is 75% copper and 25% nickel. If a mass of nickel and copper weighing 80 pounds is 90% copper, how many pounds of nickel must be added to it to make it ready for coinage into 5¢ pieces?

63. Separate 200 into three such parts that the first divided by the second gives 2 for a quotient and 2 for a remainder; and the second divided by the third gives 4 for a quotient and 1 for a remainder.

64. Simplify $(3\sqrt[4]{\tfrac{1}{9}})^2$.

65. Extract the square root of $x^2 + \frac{1}{x^2} + 2 + \frac{1}{x^4} - \frac{2}{x^3} - \frac{2}{x}$.

66. Find that number which, when divided by 3, is equal to one quarter of the sum of itself and 24.

67. Divide $2x^{\frac{9}{2}}y^{-3} - 5x^{\frac{7}{2}}y^{-2} + 7x^{\frac{5}{2}}y^{-1} - 5x^{\frac{3}{2}} + 2x^{\frac{1}{2}}y$ by $x^3y^{-3} - x^2y^{-2} + xy^{-1}$.

68. In the year 1910 the record for the baseball throw was 426 ft. 6 in., which was 8 ft. $2\frac{3}{4}$ in. more than 17 times the record for the running long jump. What was the latter record?

69. Show that $(n-1)t^2(t^2+a^2)^{\frac{n-3}{2}} + (t^2+a^2)^{\frac{n-1}{2}}$ reduces to $(nt^2+a^2)\,(t^2+a^2)^{\frac{n-3}{2}}$.

70. Solve the following equations for x and y:

$$ax + by = 1, \qquad bx - ay = 1.$$

71. Simplify $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.

72. Multiply $\sqrt[3]{\dfrac{a}{bc}}$ by $\sqrt{\dfrac{a}{bd}}$.

73. Define a literal equation. Quadratic equation. Root of an equation. Absolute term. Degree of an equation.

74. A baseball player has been to the bat 150 times in a given season and made an average of .280 hits. How many more times will he need to bat to bring his average up to .375, provided that the number of base hits he makes in the future equals half the number of times he bats?

75. Solve $\dfrac{3}{x^3-8} + \dfrac{2x+5}{2x^2+4x+8} - \dfrac{1}{x-2} = 0$.

76. Simplify the product of $(ayx^{-1})^{\frac{1}{2}}$, $(bxy^{-2})^{\frac{1}{3}}$, and $(y^2a^{-2}b^{-2})^{\frac{1}{4}}$.

77. Find the numerical value of the following expression when

$$a = 5, \quad b = 3, \quad c = -1, \quad d = -2, \quad \text{and } x = 0$$

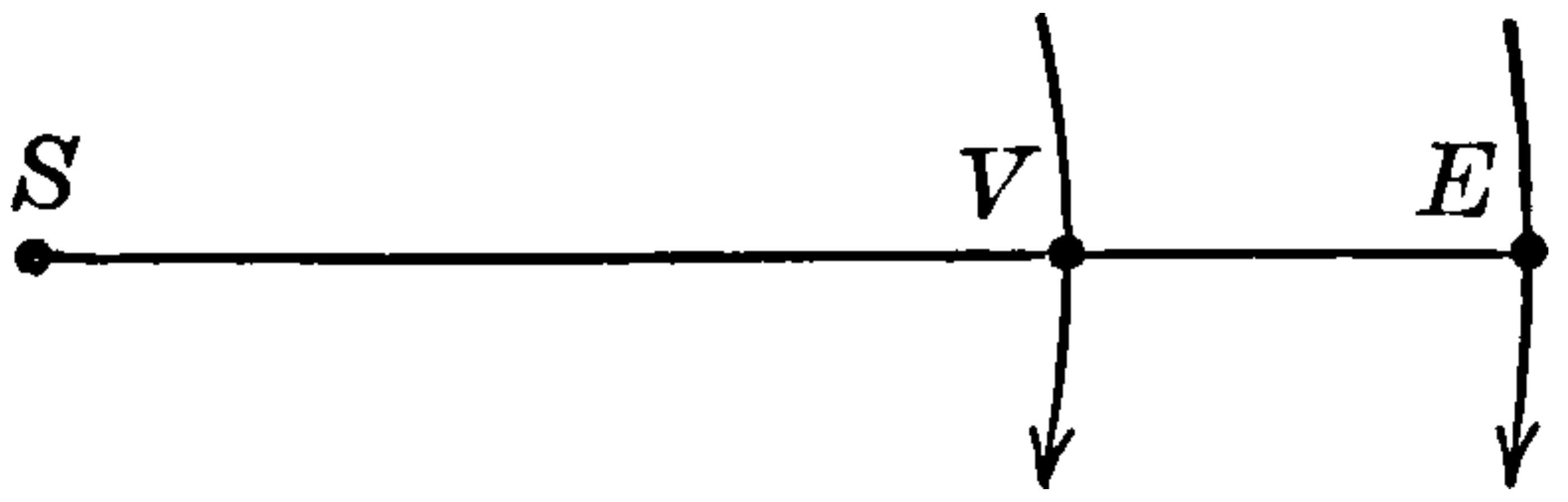
$$\frac{2\sqrt{3+2d+a}}{3\sqrt{a+b-cx-c}} + \frac{(3c-d)x}{7ad - \sqrt{abc}}$$

78. Simplify $\dfrac{\left\{\sqrt{\dfrac{p+\sqrt{p^2-q}}{2}} + \sqrt{\dfrac{p-\sqrt{p^2-q}}{2}}\right\}^2}{p+\sqrt{q}}$

79. Reduce $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ to an equivalent fraction having a rational denominator, and find its value to two decimal places.

80. Find the value of $x^2 - 6x + 14$, if $x = 3 + \sqrt{-5}$.

81. The planet Venus is said to be in *conjunction* when it is in a line between the earth and the sun. If it takes Venus 225 days to make one revolution about the sun, how long is the interval between two successive conjunctions of Venus?



82. The interval between two successive conjunctions of the planet Mercury is 116 days. How long does it take Mercury to make one revolution about the sun?

83. Simplify $\dfrac{\dfrac{x-y}{2y+x}+\dfrac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2y}{5x+7y}\right)^{-1}}$.

84. Solve $\dfrac{3x+2}{x^2+x}+\dfrac{2(x-5)}{1-x^2}-\dfrac{x-3}{x^2-x}=0$.

85. Multiply $\sqrt[4]{m^{-3}}-\sqrt{m^{-1}}\sqrt[4]{n}+\sqrt[4]{m^{-1}}\sqrt{n}-\sqrt[4]{n^3}$ by $m^{-\frac{1}{4}}+n^{\frac{1}{4}}$.

86. The natural water-power of the United States is 75,000,000 H. P. This is 5,000,000 H. P. more than 10 times the water-power of Niagara Falls. Find the latter. Make up and work a similar example concerning the fully developed water-power of the United States, if the latter is 230,000,000 H. P.

87. Simplify $\left[(a^{p+q})^{p-q}\,(a^{q^3})^{\frac{1}{q}}\right]^{\frac{1}{p^3}}$.

88. Find the values of x, y, and z which satisfy the simultaneous equations $x + 2y = 3$, $3y + z = 2$, and $2z + 3x = 1$.

89. If a number of two digits is divided by the sum of the two digits, the quotient is 4. If the digits are interchanged, the resulting number will be greater than the original number by 36. Find the number.

90. Factor $x^2 - 2ax - b^2 + 2ab$.

91. In $a^2(x^2 - yz)^{-\frac{2}{3}}$ introduce a^2 into the parenthesis without changing the value of the expression.

92. Given $V = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$, $\pi = \frac{22}{7}$, $R = 21$, and $r = 14$, find V in the shortest way.

This example illustrates the utility of what algebraic principle?

93. If $y = \frac{a + 2t}{b - at}$, find the value of t in terms of the other letters.

94. If ether boils at a temperature of $35\frac{5}{9}°$ C, at what temperature on the Fahrenheit scale will it boil?

95. Extract the square root of $7 + 4\sqrt{3}$. Of $3 + \sqrt{5}$. Of $2a + 2\sqrt{a^2 - b^2}$.

96. A sinking fund is a fund accumulated to meet a debt by setting aside a certain sum, the sum set aside to accumulate by compound interest. If

C = number of dollars in the debt, s = sum set aside annually,
n = number of years, r = rate of interest,

then it may be shown that

$$C = \frac{s[(1 + r)^n - 1]}{r}$$

If a city wishes to take up $2,500,000 worth of bonds at the end of 4 years, how much must it set aside each year, the rate of interest being 5%?

97. Solve $2 + \sqrt{x} - \sqrt{2x + 7} = 0$.

98. Simplify

$$\frac{(a-b)^{-1}(b-c)^{-1} + (b-c)^{-1}(c-a)^{-1} + (c-a)^{-1}(a-b)^{-1}}{(a-b)(b-c) + (b-c)(c-a) + (c-a)(a-b)}.$$

99. Rationalize the denominator of $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{c}}$ and find the value of the result, when $a = \frac{4}{5}$, $b = 20$, and $c = 5$.

100. Find the value of $\sqrt{a^3b^{-\frac{2}{3}} - 4a^{-\frac{3}{2}}b^{-\frac{1}{3}} + 6 - 4a^{-\frac{3}{2}}b^{\frac{1}{3}} + a^{-3}b^{\frac{2}{3}}}$.

101. Simplify $\frac{x^2 - (y - z)^2}{(x + z)^2 - y^2} + \frac{y^2 - (x - z)^2}{(x + y)^2 - z^2} + \frac{z^2 - (x - y)^2}{(y + z)^2 - x^2}$.

102. Simplify $\frac{1 + n - n^3 - n^4}{1 - a^2} \div \frac{n^2 - 1}{a^2 - 1}$.

103. A sum of $1050 is divided into two parts and invested. The simple interest on the one part at 4% for 6 years is the same as the simple interest on the other at 5% for 12 years. Find how the money is divided.

104. Simplify $3\sqrt{\frac{5}{2}} + \sqrt{40} + \sqrt{\frac{2}{5}} - \frac{1}{\sqrt{10}}$.

105. Factor $\frac{x^2}{y^2} - 3\frac{y^2}{x^2} + 2.$

106. Solve $(x - a)(b - c) + (x - b)(c - a) = 0$ for x.

107. Simplify $\dfrac{1}{a - \dfrac{1}{b - \dfrac{1}{c}}}.$

108. Find the values of x and y which satisfy simultaneously the following equations: $\frac{2}{x} + \frac{1}{y} = 2$, $\frac{3}{x} = 5 + \frac{2}{y}.$

109. Solve for x, y, and z: $ax + by = c$, $cx + az = b$, $bz + cy = a$.

110. Find the greatest common divisor of $a^2x^2 - 2acxz - b^2y^2 + c^2z^2$ and $a^2x^2 + 2abxy + b^2y^2 - c^2z^2$.

111. Solve $\dfrac{1}{x}\left(\dfrac{1}{\dfrac{1}{x} + 1} - \dfrac{1}{\dfrac{1}{x} - 1}\right) + \dfrac{1 + x\sqrt{2}}{1 - x\sqrt{3}} = \dfrac{3x^2 - 2x - 3}{1 - x^2}.$

112. Simplify $[(x + y)^5 + (x - y)^5][(x + y)^5 - (x - y)^5]$.

113. At what time between 7 and 8 P. M. are the hands of a clock opposite each other?

114. Solve for x, y, and z: $x + y = xy$, $2x + 2z = xz$, $3z + 3y = yz$.

115. The indicated horse-power of a steam engine is found by use of the formula

$$\text{H. P.} = \frac{plan}{33{,}000}$$

where p = average steam pressure in pounds per square inch,
l = length of the piston stroke in feet,
a = piston area in square inches,
n = number of revolutions per minute.

In an engine whose piston area is 402.12 sq. in., and the length of whose stroke is $2\frac{1}{2}$ ft., find the indicated horse-power (to the nearest unit), when the steam pressure is 40 pounds per square inch, and the number of revolutions is 30 per minute.

Also solve the above formula for n. For p.

116. Write a statement of the advantages in representing numbers by letters in algebra.

117. Which is greater, $2\sqrt[3]{9}$ or $3\sqrt{3\sqrt{2}}$?

118. Free the following equation from radicals and find the value of x when $q = 0$:

$$x = -\tfrac{1}{2}q + \sqrt{-\tfrac{3}{4}q^2 + \sqrt{q^4 + r^4}}.$$

119. Solve $\quad \dfrac{5}{3x} - \dfrac{2}{5y} = 7, \quad \dfrac{7}{6x} - \dfrac{1}{10y} = 3.$

120. Factor $ac(a - c) - ab(a - b) - bc(b - c)$.

121. If 19 pounds of gold and 10 pounds of silver each lose one pound when weighed in water, find the amount of each in a mass of gold and silver that weighs 106 pounds in air and 99 pounds in water.

122. From $[m(3m-p)-2n(4n-3p)]x + [m(p-m)-p(2n+p)]y$ take $3\left[p\left(2n-\frac{3p}{2}\right)-\frac{p}{2}(2m-3p)\right]x-[p(p-m)+2n(2n+p)]y.$

123. Show that $\dfrac{x}{x^{\frac{1}{3}}-1} - \dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}+1} - \dfrac{1}{x^{\frac{1}{3}}-1} + \dfrac{1}{x^{\frac{1}{3}}+1} = x^{\frac{2}{3}} + 2.$

124. A man having 10 hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot at the rate of 3 miles an hour. Find the distance he rode.

125. Find the value of $1^{-2} + 1^{-\frac{1}{3}} + 1^0 - 1^n$.

126. Find the value of x which satisfies the equation

$$\frac{3x-4}{x^2-3x+2} - \frac{x^2+1}{x-1} = \frac{2x+3}{x-2} - (x+3).$$

127. Find the numerical value of $(5.1)^{\frac{3}{2}}$ to 4 decimal places.

128. Simplify $\dfrac{1}{x+\sqrt{x^2-1}} + \dfrac{1}{x-\sqrt{x^2-1}}\cdot$

129. Simplify $\left(\dfrac{x^4+x^3-x-1}{1-y^2}\right)\left(\dfrac{y^2-1}{x^2-x}\right)\left(1-\dfrac{1}{1-\dfrac{1}{x}}\right)\cdot$

130. Does $\sqrt{a^2b^2} = ab$? Does $\sqrt{a^2+b^2}$ equal $a + b$?

131. Solve for x and y: $\dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a, \ \dfrac{x-y}{4ab} = 1.$

132. Find the square root of the product of $x^2 - 1$, $x^2 - 3x + 2$, and $x^2 - x - 2$.

133. Find the value of $\dfrac{1 - 2x + x^2}{1 + 2x + x^2}$ when $x = \sqrt{2} - 1$.

134. Rationalize the denominator of $\dfrac{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}}{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}}$.

135. Simplify $(2\sqrt{-1})^4$.

136. An automobile ran 100 miles in 4 hr. and 30 min. In the second half of the journey the speed was 5 miles per hour greater than in the first half. Find the speed in each half.

137. Simplify $\{x^{\frac{3}{2}}y^{-\frac{5}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) \div (x^{\frac{3}{2}} + y^{\frac{3}{2}})\}\,(x + y - x^{\frac{1}{2}}y^{\frac{1}{2}})$ and find its value when $x = 18$ and $y = 2$.

138. Solve $\dfrac{a}{bx} + \dfrac{b}{ay} = a + b$, $\dfrac{b}{x} + \dfrac{a}{y} = a^2 + b^2$.

139. Divide 1 by $\sqrt{x} - 1$ to four terms and extract the square root of the quotient to three terms.

140. Find the highest common factor of $x^4 + x^2y^2 + y^4$ and $x^3 + y^3$.

141. Reduce $\dfrac{A}{x} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2} + \dfrac{D}{(x + 1)^3}$ to a common denominator and arrange the terms of the numerator according to the ascending powers of x.

142. By finding the value of t in the first equation and substituting in the second, elimate t between the equations $v = u + gt$ and $s = ut + \frac{1}{2}gt^2$. Hence find s when $g = 32$, $v = 10.4$, and $u = 2.2$.

143. From $\dfrac{3}{5 - 2\sqrt{3}}$ subtract $\dfrac{1}{3 + \sqrt{3}}$ and express the result as a fraction having a rational denominator.

144. Simplify $x - \dfrac{1}{x^2 + x - \dfrac{x^3}{1 - x} - \dfrac{1}{1 - \dfrac{1}{x}}}$.

145. If $\sqrt{2x^3} = \sqrt{5y^2}$ and $x = \sqrt{2y}$, find x and y.

146. Simplify $2\sqrt{24} \cdot 3\sqrt[4]{18} \cdot 4\sqrt[6]{24}$.

147. Extract the square root of 1.672 to four decimal places.

148. Solve for x: $\dfrac{\sqrt{x}-8}{\sqrt{x}-6}=\dfrac{\sqrt{x}-4}{2+\sqrt{x}}$.

149. Divide $\sqrt[3]{x^4}-4xy+4y\sqrt[3]{x^2}+4y^2$ by $\sqrt[3]{x^2}+2\sqrt{xy}+2y$.

150. Simplify $(6x^2-6)\left(\dfrac{3x-3}{2x+2}\right)^{\frac{1}{2}}$.

151. A gave B as much money as B had; then B gave A as much money as A had left; finally A gave B as much money as B then had left. A then had $16 and B, $24. How much had each originally?

152. Simplify $16+\left\{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}-2\dfrac{x^2-a^2}{x^2+a^2}\right\}^2$.

153. Factor $(a-b)x^2+2ax+(a+b)$.

154. A man and two boys do a piece of work in 24 da. which could have been done in 12 da. by three men and one boy. How long would it take two men and two boys to do the work?

155. Solve $\dfrac{x-\frac{5}{3}}{\frac{5}{3}(x-1)}+\dfrac{\frac{7}{3}-x}{\frac{7}{3}(1+x)}=\dfrac{1}{35\left(1-\dfrac{1}{x^2}\right)}+\dfrac{1}{7}$.

156. Collect in the shortest way

$$\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}.$$

157. Find the value of $\dfrac{4\sqrt{12.5}-2\sqrt{.5}}{6\sqrt{18}}$.

158. Simplify $\dfrac{a^2}{b^2}\sqrt[n]{\dfrac{b^{n+1}}{a^{n-1}}}$.

159. Solve $\dfrac{1}{\sqrt{a-x}+\sqrt{a}}+\dfrac{1}{\sqrt{a-x}-\sqrt{a}}=\dfrac{\sqrt{a}}{x}$.

160. Show that the following polynomials do not have a common factor: $x^2+2x-8,\ x^3+x^2-3x+1$.

161. Solve $\dfrac{x^n-2x^{n-1}}{2x-4}+\dfrac{x^n}{x-2}-\dfrac{3x^n}{2x-2}=0$.

162. Rationalize the denominator of $\dfrac{\sqrt{1-\frac{1}{3}\sqrt{2}}}{\sqrt{1-\dfrac{\sqrt{5}}{3}}}$.

163. Extract the square root of

$$\frac{a^2x^2}{9}+\frac{b^2}{25}+\frac{9x^4}{4}-ax^3-\frac{3bx^2}{5}+\frac{2abx}{15}.$$

164. Extract the square root of $x^4+4x^3+8x^2+8x-21$ until a numerical remainder is obtained, and thus show that the original expression equals $(x^2+2x+2)^2-25$. Hence obtain the factors of the original expression. Treat in like manner $y^4+6y^3+11y^2+6y-8$.

EXERCISE 158

Review, Beginning with Quadratics

Solve:

1. $x^2+\frac{1}{x^2}=a^2+\frac{1}{a^2}$.

2. $x^2+4x-2\sqrt{3}=0$.

3. $x^2+1.8x-11.5=0$.

4. $.3x^2+.5x=.2$.

5. Find the two values of x which satisfy the equation

$$\sqrt[4]{x-1}+2\sqrt{x-1}-1=0.$$

6. By writing a numerical quadratic equation, as $(x-2)^2=9$, in the form $(x-2)^2-9=0$, show that the solution by completing the square may be reduced to the factorial solution.

7. Solve $x+\frac{1}{x}=1+\cfrac{1}{1-\cfrac{4b^2}{a^2+3b^2}}$.

8. By letting a, b, c, etc., have special values, convert $ax^2+bxy+cy^2+dx+ey+f=0$ into

(1) a homogeneous equation of the second degree.
(2) a symmetrical equation of the second degree.
(3) a homogeneous symmetrical equation of the second degree.

Solve:

9. $\frac{1}{x^2}+\frac{1}{y^2}=13$.
$6xy=1$.

10. $\sqrt{x+y}=3$.
$xy=8$.

11. $xy=80$.
$\frac{1}{x}-\frac{1}{y}=\frac{1}{5}$.

12. $x^2+xy+x=14$.
$y^2+xy+y=28$.

13. $\frac{1}{x^2}+\frac{1}{y^2}=a.$

$\frac{1}{x^2}+\frac{1}{z^2}=b.$

$\frac{1}{y^2}+\frac{1}{z^2}=c.$

14. $x^2y+xy^2=6.$

$3x^2y^2+8xy=3.$

15. $\frac{1}{x}+\frac{1}{y}=2.$

$xy+\frac{1}{x}+\frac{1}{y}=3\frac{1}{3}.$

16. $x+y+\sqrt{xy}=14.$

$\sqrt{xy}\,(x+y)=40.$

17. $\sqrt{x^2+7}+y=6.$

$\sqrt{x^4+22y^2}+x^2=22.$

18. $\sqrt{x+y}+\sqrt{x-y}=4.$

$x^2-y^2=9.$

19. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}.$

$x+y=10.$

20. $\sqrt{xy}-\sqrt{x-y}=11.$

$\sqrt{x-y}\sqrt{xy}=60.$

21. Write the equations whose roots are $1 \pm i$. Also $\frac{1}{2} \pm \frac{1}{2}i$.

22. Find by inspection the sum of the roots of $3x^2-2x+1=0$. Find also the product of the roots. Verify your result by solving the given equation. About how much shorter is the first process than the second?

23. What must be added to each of the terms of $a^2 : b^2$ to make the resulting ratio equal to $a : b$?

24. What number must be subtracted from each of the numbers, 9, 12, 15, and 21, so that the remainder shall form a proportion?

25. If a box car $36' \times 8\frac{1}{2}' \times 8'$ has a capacity of 60,000 lb., by how much must the length be increased to make the capacity 100,000 lb.?

26. When $z=25$, solve the following system of proportions:

$$x:y:z:w=3:4:5:6.$$

27. The rates of two boys traveling on bicycles are as p to q. If the first boy rides a miles in a given time, how far does the other boy travel in the same time?

28. For what value of x will the ratio $x^2-x+1 : x^2+x+1$ be equal to $3 : 7$?

29. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{p}{q}$, prove $\frac{p}{q}=\frac{5a+3c-2e}{5b+3d-2f}$.

30. If $a:b=c:d$, show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

31. If $a : b = c : d$, and a is not equal to b or c, prove that it is impossible to find a number x (other than zero) such that $a - x : c - x = b - x : d - x$.

32. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$, prove that $\frac{\sqrt{px^2 + qy^2 + rz^2}}{\sqrt{pa^2 + qb^2 + rc^2}} = k$.

33. Find the sum of 30 terms of the A. P. 3, 5, 7, ... by the addition of successive terms. Now find this sum by the use of one of the formulas of Art. 278. Compare the amount of work in the two processes.

34. Prove that the differences between the squares of successive integers forms an A. P.

35. Prove that equimultiples of the terms of an A. P. form another A. P.

36. Obtain a formula for the nth term of the A. P. 9, 7, 5, ... Also for the $n + 2$d term.

37. If the hours of the day were numbered from 1 to 24, how many times would a clock strike in striking the hours during one day?

38. Show that the sum of n consecutive integers is divisible by n, if n is odd but not if n is even.

39. Find the sum of n terms of $1, 2b, 4b^2, 8b^3, \ldots.$

40. If each stroke of an air pump removes $\frac{2}{3}$ of the air in a receiver, what fraction of the air will be left in the receiver after 10 strokes?

41. Find a G. P. in which the sum of the first two terms is $2\frac{2}{3}$ and the sum to infinity is $4\frac{1}{6}$.

42. Find the 7th term of $\left(2x^{\frac{1}{2}} - \frac{x^{-\frac{3}{2}}}{2}\right)^{15}$ by finding all the terms up to the 7th. Now find the 7th term by the method of Art. 294. Compare the amount of work in the two processes.

43. Expand $(\sqrt{x+1} - \sqrt{x-1})^4$.

44. Find the 98th term of $(3a - 2b)^{100}$.

45. Find in a short way the sum of the coefficients of the terms in the expansion of $(2a - b)^8$. Of $(2a + b)^8$.

46. Find the two middle terms of $\left(2\sqrt{x} - \frac{1}{2x^{\frac{1}{3}}}\right)^{11}$.

47. In the expansion of $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^8$, find the coefficient of x^4.

48. Find the ratio between the sixth term in the expansion of $\left(1+\frac{3x}{2}\right)^{10}$ and the fifth term in the expansion of $\left(1+\frac{3x}{2}\right)^9$.

49. Solve $\sqrt{x+1}+\sqrt{x-2}=\sqrt{2x+3}$.

50. Find the sum of all positive integers of three digits which are divisible by 9.

51. What is meant by an irrational root of an equation? By an imaginary root?

52. Solve $x^2+x-4ax+3a^2-5a-2=0$.

53. Solve $x^2y-xy^2=12,\ x^3-y^3=63$.

54. Solve $\sqrt[4]{x^2+12}+\sqrt{x^2+12}=6$.

55. The sum of 5 terms of an A. P. is – 5, and the 6th term is – 13. What is the common difference?

56. What is the ratio of the mean proportional between a and b to the mean proportional between a and c?

57. Form an equation whose roots are $-\frac{1\pm 2\sqrt{5}}{3}$.

58. A boat crew rowing at half their usual speed row 3 mi. down stream and back again in 2 hr. and 40 min. At full speed they can go over the same course in 1 hr. 4 min. Find in miles per hour the rate of the crew and of the current.

59. Solve $x^2+xy+y^2=1,\ 2x^2+3xy+4y^2=3$.

60. Show that the sum of the squares of the roots of the equation $x^2-5x+2=0$ is 21.

61. The mean annual rainfalls at Phoenix (Ariz.), Denver, Chicago, and New Orleans are 7.9 in., 14 in., 34 in., and 57. 4 in. respectively. By how much do these numbers differ from the corresponding terms in a G. P. whose first term is 7.9 in., and whose ratio is 2?

62. Solve in the shortest way $\frac{1}{x-8}+\frac{1}{x-6}+\frac{1}{x+6}+\frac{1}{x+8}=0$.

63. If $a:b=c:d=e:f$, show that $a^3+c^3+e^3:b^3+d^3+f^3=ace:bdf$.

64. Solve $\sqrt{x}+\sqrt{y}=5,\ x+y=13$.

65. By use of the binomial theorem find the ratio of the 5th term to the 7th term in the expansion of $(1 - \sqrt{2x})^7$.

66. Solve $\frac{x^2}{x+1} + \frac{x+1}{x^2} = \frac{5}{2}$.

67. Derive the formulas for the nth term and for the sum of n terms of a geometrical progression in terms of the first term and the common ratio.

68. Solve $n^2(x^2 + 1) = a^2 + 2n^2x$.

69. Which term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, etc., is 8?

70. Solve $x^2 - 3x - 6\sqrt{x^2 - 3x - 3} + 2 = 0$.

71. Show that the roots of the equation $x^2 + ax - 1 = 0$ are real and unequal for any real value of a.

72. Solve $\left(x + \frac{1}{x}\right)^2 = 4 + \left(1 - \frac{1}{x}\right)\left(1 + \frac{1}{x}\right)$.

73. Find four numbers in A. P. such that the sum of the first and third shall be 18, and the sum of the second and fourth shall be 30.

74. Find the G. P. whose sum to infinity is 4 and whose second term is $\frac{3}{4}$.

75. A rectangular park is 100 rods long and 80 rods wide. By what per cent must its dimensions be increased in order that its area shall be doubled?

76. The difference between the reciprocals of two consecutive numbers is $\frac{1}{30}$. Find the numbers.

77. Find the sum to infinity of $-3 + \frac{1}{2} - \frac{1}{12} \cdots$

78. Given $K = \pi R^2$, and $C = 2\pi R$, eliminate R and find K in terms of C.

79. Given $S = \pi RL$ and $T = \pi R(R + L)$, eliminate R and find T in terms of S and L.

80. By use of an A. P. find the sum of all the numbers between 1 and 207 which are divisible by 5.

81. A man sold a horse for $96 and in doing so gained as many per cent as the horse cost him dollars. What did the horse cost him?

82. Solve $x + y + \sqrt{x + y} = 20$, $xy = 63$.

83. Given l = distance in feet between two adjacent supports of a trolley wire,

s = sag of the wire in feet,
t = tension of the wire in pounds,
w = weight of wire in pounds,
l' = actual length of wire between two adjacent supports,

and (a) $s = \frac{wl^2}{8t}$ . . . and (b) $l' = l + \frac{8s^2}{3l}$,

(1) Find the value of l in equation (a). Also in equation (b).
(2) Eliminate l between the two equations.

84. Expand $(2\sqrt{3} - \frac{1}{2}\sqrt{-2})^5$ and simplify.

85. The sum of the first seven terms of a G. P. is 635, and the ratio is 2. What is the fourth term?

86. Two boys start on bicycles at the vertex of a right angle and ride along its sides at the rate of 6 and 8 miles per hour, respectively. How many hours will it be before they are 100 miles apart?

87. Determine by inspection the roots of the equation

$$ax(bx - 2)(x^2 - 9) = 0.$$

88. If α and β are the roots of the equation $px^2 + qx + r = 0$, find the values of $\alpha + \beta$, $\alpha - \beta$, and $\alpha\beta$ in terms of p, q, and r.

89. A man finds that it takes him 2 hours less to walk 24 miles, if he increases his speed 1 mile per hour. What is his usual rate?

90. If $a : b = b : c$, prove that $a + b : b + c = b^3 : ac^2$.

91. Insert four geometric means between 160 and 5.

92. How many terms of the A. P. 42, 39, 36, ... must be taken to make 315?

93. Express the repeating decimal .3232 . . . . as a fraction.

94. Solve $(x^{-2} + \frac{1}{4})^{-3} = 27$.

95. Solve $9x^2 + 25y^2 = 148$, $5xy = 8$.

96. The first term of a geometrical series is 2 and the sum of the fourth term and three times the second term is equal to four times the third term. Find the series.

97. Solve $(x^{\frac{1}{2}} - x)(x + 2) = 0$.

98. Solve $\frac{x^2}{y} - \frac{y^2}{x} = 8\frac{2}{3}$, $x - y = 2$.

99. If $\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a-b}$, show that $x - y + z = 0$.

100. Solve $x(x - y) = 0$, $x^2 + 2xy + y^2 = 9$ by the factorial method as far as possible.

101. In the same way solve
$(y + 3x - 7)y = 0$, $(y + x - 3)(y + 2x - 4) = 0$.

102. Solve $x^{\frac{1}{3}} + 3x^{-\frac{1}{3}} = 4$.

103. The hypotenuse of a right triangle is 20. The sum of the other two sides is 28. Find the length of the sides.

104. Solve $x^2y^2 - 10xy + 24 = 0$, $x + y = 5$.

105. The sum of the first seven terms of an A. P. is 98, and the product of the first and seventh terms is 115. Find the common difference.

106. Solve $x^2 + xy + y^2 = 133$, $x - \sqrt{xy} + y = 7$.

107. Find the sum of the odd integers between 0 and 200. How many of these are not divisible by 3?

108. Find the values of x and y which will satisfy the following:

$$x + \frac{1}{y} = 1,\ y + \frac{1}{x} = 4.$$

109. In an A. P., given $a = \frac{1}{2}$, $l = -2\frac{1}{2}$, $s = -4$, find n and d.

110. If the speed of a train should be lessened 4 miles an hour, the train would be half an hour longer in going 180 miles. Find the rate of the train.

111. Plot the graphs of the following system of equations: $x^2 + y^2 = 4$, $3x - 2y = 6$. From the graphs find the approximate values of x and y that satisfy both equations.

112. Solve $9x - 3x^2 + 4\sqrt{x^2 - 3x + 5} = 11$.

113. The sum of an infinite G. P. is 4 and the first term is 6. Find the ratio and the sum of 4 terms.

114. Solve $\sqrt{x} + \sqrt{3 - \sqrt{3x + x^2}} = \sqrt{3}$.

115. What is meant by an extraneous root of an equation? Give an example of an extraneous root.

116. At his usual rate a man can row 15 miles down stream in 5 hours less than it takes him to return. If he could double his rate his time down stream would be only 1 hour less than his time up stream. What is his rate in still water?

117. Given $S = 4\pi R^2$, and $V = \frac{4}{3}\pi R^3$, eliminate R and find V in terms of S.

118. Solve $5x^{\frac{2}{5}} - 3x^{\frac{1}{5}} - 14 = 0$.

119. If the 6th term of an A. P. is 9, and the 16th term is $22\frac{1}{3}$, find the 25th term and the sum of 30 terms.

120. What two numbers, whose difference is h, are to each other as $a : b$?

121. If $\frac{x}{a-b} = \frac{y}{b-c} = \frac{z}{c-a}$, find the value of $x + y + z$.

122. What distance is passed over by a ball which is thrown 60 feet vertically upward and at every fall rebounds $\frac{1}{3}$ the distance from which it fell?

123. Solve $x^2 + y^2 + 2(x + y) = 12$, $xy - (x + y) = 2$.

124. What number added to both numerator and denominator of $\frac{a}{b}$, and subtracted from both numerator and denominator of $\frac{c}{d}$, will make the results equal?

125. Find the tenth term of $-\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \ldots$ and the sum of the series to ten terms.

126. Solve $x^2 - 2\sqrt{7}x + 2 = 0$.

127. Find in a short way the sum of the coefficients in the expanded form of $(2x - \sqrt{y})^{20}$.

128 If $a : b = b : c = c : d$, show that $b + c$ is a mean proportional between $a + b$ and $c + d$.

129. If a, b, c, and d are in A. P., show that $a + d = b + c$.

130. Solve $x + y + \sqrt{x + y} = 12$, $x - y + \sqrt{x - y} = 2$.

131. Solve $(x + 1 + x^{-1})(x - 1 + x^{-1}) = 5\frac{1}{4}$.

132. If a boy runs 100 yd. in 10 sec. how much does his velocity differ from a mean proportional between the velocity of a man walking 4 mi. per hour and an express train going 60 mi. per hour?

133. The following table gives the normal or average height of a boy and girl at different ages:

Age in years	3	6	9	12	15	18	21
Height of boy	2′11″	3′8″	4′2″	4′7½″	5′2¾″	5′6¾″	5 8¼
Height of girl	2′11″	3′7″	4′2″	4′9″	5′1½″	5′3¼″	5′3¾″

Graph the above facts as two graphs on one diagram.

From these graphs determine as accurately as you can the normal height of a boy and of a girl at 10 years of age. At 14 years.

134. The sum of the first ten terms of a G. P. is equal to 244 times the sum of the first five terms, and the sum of the fourth and sixth terms is 135. Find the first term and the common ratio.

SUG. Show that $r^{10} - 1 = 244(r^5 - 1)$, etc.

135. Insert between 1 and 21 a series of arithmetical means such that the sum of the last three shall be equal to 48.

136. Find the coefficient of x^n in the expansion of $\left(x^2 - \frac{a^3}{x}\right)^{2n}$.

137. Prove that either root of the quadratic equation $x^2 - q = 0$ is a mean proportional between the roots of $x^2 + px + q = 0$.

138. Simplify $(\sqrt{a+b} + \sqrt{a-b})^6 + (\sqrt{a+b} - \sqrt{a-b})^6$.

139. Solve $x + y = 5 - xy$, $x + y = \frac{6}{xy}$.

140. Find the sum of n terms of the series

$$(x - y) + \left(\frac{y^2}{x} - \frac{y^3}{x^2}\right) + \left(\frac{y^4}{x^3} - \frac{y^5}{x^4}\right) + \dots.$$

141. The formula used for determining the elevation of the outer rail of a railroad track on a curve is as follows: $E = \frac{4BV^2}{5R}$,

where E = elevation of outer rail in inches
B = width of the track in feet
R = radius of the curve in feet
V = maximum speed in miles per hour of a train taking the curve.

Find E when B = 4 ft. 8½ in., R = 425 ft., V = 20 mi. per hour. Also when V = 60 mi. per hour.

142. Solve the formula for V. From this result determine the maximum speed at which a train can take the track when E = 5 in.

MATERIAL FOR EXAMPLES

FORMULAS

Formulas used in the following subjects may be made the basis for numerous examples.

I. ARITHMETIC

$$p = br$$
$$i = prt$$
$$a = p + prt$$

$$C = \frac{s[(1 + r)^n - 1]}{r}$$

II. GEOMETRY

$$K = \tfrac{1}{2}bh$$
$$K = \tfrac{1}{4}a^2\sqrt{3}$$
$$K = \tfrac{1}{2}h(b + b')$$
$$C = 2\pi R$$
$$K = \pi R^2$$
$$K = \pi RL$$
$$S = 4\pi R^2$$

$$T = \pi R(R + L)$$
$$T = 2\pi R(R + H)$$
$$V = \pi R^2 H$$
$$V = \tfrac{1}{3}\pi R^2 H$$
$$V = \tfrac{4}{3}\pi R^3$$
$$S = \frac{\pi R^2 E}{180}$$

$$K = \sqrt{s(s - a)(s - b)(s - c)}$$
$$V = \tfrac{1}{3}H(B + b + \sqrt{Bb})$$

III. PHYSICS

$$v = gt$$
$$s = \tfrac{1}{2}gt^2$$
$$s = \frac{v^2}{2g}$$
$$s = vt + \tfrac{1}{2}gt^2$$

$$F = \frac{mv^2}{r}$$
$$E = \frac{mv^2}{2}$$

$$E = \frac{wv^2}{2a}$$

$$e = \frac{4Pl^3}{bh^3m}$$

$$t = \pi\sqrt{\frac{l}{g}}$$

$$C = \frac{E}{R}$$

$$R = \frac{gs}{g + s}$$

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{p}$$

$$H = .24C^2Rt$$

$$E = \frac{4n^2l^2w}{g}$$

$$C = \tfrac{5}{9}(F - 32)$$

IV. ENGINEERING

$\text{H. P.} = \dfrac{plan}{33{,}000}$ (horse-power in an engine)

$s = \dfrac{wl^2}{8t}$ and $l' = l + \dfrac{8s^2}{3l}$ (sag in a suspended wire)

$E = \dfrac{4BV^2}{5L}$ (elevation of outer rail on a curve)

$W = \dfrac{B \times D^2}{L}k$ (weight a beam will support)

$L = \dfrac{.0045(P-t)\ (T-t)C}{P}$ (length of hot-water pipe to heat a house)

$T = \dfrac{D^2PL}{W}$ (tractive force of a locomotive)

$G = \dfrac{\sqrt{(15D)^5H}}{L}$ (no. gal. water delivered by a pipe)

$D = \sqrt{\dfrac{G}{.34LN}}$ (diameter of a pump to raise a given amount of water)

$D = \sqrt[3]{\dfrac{W}{.5236(A-G)}}$ (diameter of balloon to raise a given weight)

IMPORTANT NUMERICAL FACTS

AREAS

	Sq. Mi.
Rhode Island	1250
New Jersey	7815
New York	49,170
Texas	265,780
United States	3,025,600
North America	6,446,000
Land surface of earth	51,238,800
Great Britain and Ireland	121,371
France	207,054
Europe	3,555,000

ASTRONOMICAL FACTS

Planet	Diameter in Miles	Distance from Sun in Million Miles	Time of Revolution about Sun	Synodic Period in Days
Mercury	3030	36	88 da.	116
Venus	7700	67.2	225 da.	584
Earth	7918	92.8	365 da.	
Mars	4230	141.5	687 da.	780
Jupiter	86,500	483.3	11.86 yr.	399
Saturn	73,000	886	29.5 yr.	378
Uranus	31,900	1781	84 yr.	369
Neptune	34,800	2791	165 yr.	367

Sun's diameter	866,400 mi.
Moon's diameter	2162 mi.
Moon's distance	238,850 mi.

Distance of nearest fixed star, 21 millions of millions of miles (or 3.6 light years).

DATES (A. D. UNLESS OTHERWISE STATED)

Rome founded	753 B. C.
Battle of Marathon	490 B. C.
Fall of Jerusalem	70
Fall of Rome	476
Battle of Hastings	1066
Printing with movable type	1438
Fall of Constantinople	1453
Discovery of America	1492
Jamestown founded	1607
Declaration of Independence	1776
Washington inaugurated	1789
Battle of Waterloo	1815
Telegraph invented	1844
First transatlantic cable message	1858
Telephone invented	1876
Battle of Manila Bay	1898

DISTANCES

From New York to	*Miles*
Boston	234
Buffalo	440
Chicago	912
Denver	1930
San Francisco	3250
Philadelphia	90
Washington	228
New Orleans	1372
Havana	1410
London	3375
San Francisco to Manila	4850
New York to San Francisco via Panama	5240
London to Bombay via Suez	6332

HEIGHTS OF MOUNTAINS

	Feet
Mt. Washington	6290
Pike's Peak	14,147
Mt. McKinley	20,464
Mt. Everest	29,002
Mt. Mitchell	6711
Mt. Whitney	14,501
Mt. Blanc	15,744
Acongua	23,802

HEIGHTS (OR LENGTHS) OF STRUCTURES

	Feet		
Bunker Hill Monument	221	Olympic	882 ft.
Washington Monument	555	Deepest shaft	5000 ft.
Singer Building (N. Y.)	612	Deepest boring	6573 ft.
Metropolitan Life Building	700	Simplon Tunnel	$12\frac{1}{4}$ mi.
		Panama Canal	49 mi.
Eiffel Tower	984	Suez Canal	100 mi.

LENGTHS OF RIVERS

	Miles		Miles
Hudson	280	Mississippi	3160
Ohio	950	Rhine	850
Colorado	1360	Amazon	3300
Missouri	3100	Nile	3400

RAINFALL (MEAN ANNUAL)

	Inches		Inches
Phoenix (Ariz.)	7.9	New York	44.8
Denver	14	New Orleans	57.4
Chicago	34	Cherrapongee (Asia)	610

RECORDS (YEAR 1910)

100-yard dash	$9\frac{3}{5}$ sec.
Quarter-mile run	47 sec.
Mile run	4 m. $15\frac{2}{5}$ sec.
Mile walk	6 m. $29\frac{1}{5}$ sec.
Running high jump	6 ft. $5\frac{5}{8}$ in.
Running broad jump	24 ft. $7\frac{1}{4}$ in.
Pole vault	12 ft. $10\frac{7}{8}$ in.

100-yard swim	$55\frac{2}{5}$ sec.
1-mile swim	23 m. $16\frac{4}{5}$ sec.
100-yard skate	$9\frac{4}{5}$ sec.
1-mile skate	2 m. 36 sec.
1 mile on bicycle	1 m. 5 sec.
1 mile in automobile	$27\frac{1}{3}$ sec.
1 mile by running horse	1 m. $35\frac{2}{5}$ sec.
1 mile by trotting horse in race	$2{:}03\frac{1}{4}$ m.
Throw of baseball	426 ft. 6 in.
Drop kick of football	189 ft. 11 in.
Transatlantic voyage (from N. Y.)	4 da. 14 h. 38 m.
Typewriting from printed copy	123 words in one minute
Typewriting from new material	6,136 words in one hour
Shorthand	187 words in one minute
Cost 1 lb. radium	$2,500,000
Corn crop per acre	$255\frac{3}{4}$ bu.
Milk from 1 cow (1 year)	27,432 lb.
Butter from cow (1 year)	1164.6 lb.

Resources (crops, etc., year 1910)

(All these figures are approximate estimates.)

Coal lands in U. S.	400,000	sq. mi.
Coal in U. S.	2,500,000,000,000	tons
Iron ore in U. S.	15,000,000,000	tons
Water-power of Niagara	7,000,000	H. P.
Natural water-power in U. S.	75,000,000	H. P.
Possible water-power in U. S. (developed by storage dams, etc.)	200,000,000	H. P.
Reclaimable swamp lands in U. S.	80,000,000	acres
Lands in U. S. reclaimable by irrigation	100,000,000	acres

National forest reserves of U. S.	168,000,000 acres
Corn crop of U. S.	3,000,000,000 bu.
Wheat crop of U. S.	700,000,000 bu.
Cotton crop of U. S.	13,000,000 bales

TEMPERATURES (FAHRENHEIT)

Normal temperature of the human body . . . . 98.7°

Ether boils at 96°
Alcohol boils at 173°
Water boils at 212°
Sulphur fuses at 238°
Tin fuses at 442°
Lead fuses at 617°
Iron fuses at 2800° (approx.)

Temperature of arc light 5400° (approx.)

Average change of temperature
below earth's surface 1° per 62 ft. (increase)
above earth's surface 1° per 183 ft. (decrease)

VELOCITIES

Wind	18 mi. per hr. (av.)
Sensation along a nerve	120 ft per sec. (av.)
Sound in the air	1090 ft. per sec. (av.)
Rifle bullet	2500 ft. per sec. (av.)
Message in submarine cable	2480 mi. per sec.
Light	186,000 mi. per sec. (approx.)

WEIGHTS

Boy 12 years old	75 lb. (av.)
Man 30 years old	150 lb. (av.)
Horse	1000 lb. (av.)
Elephant	$2\frac{1}{2}$ tons (av.)
Whale	60 tons (approx.)
1 cu. ft. of air	$1\frac{1}{4}$ oz. (approx.)
1 cu. ft. of water	62.5 lb.

Specific Gravities

Air	$\frac{1}{800}$	Stone (average)	2.5
Cork	.24	Aluminum	2.6
Maple wood	.75	Glass	2.6–3.3
Alcohol	.79	Iron (cast)	7.4
Ice	.92	Iron (wrought)	7.8
Sea water	1.03	Lead	11.3
Water	1	Gold	19.3
Clay	1.2	Platinum	21.5

Miscellaneous

Heart beats per minute — Frog	10
Man	72
Bird	120
Smallest length visible to unaided eye	$\frac{1}{250}$ inch
Smallest length visible by aid of microscope	$\frac{1}{125,000}$ inch
Accuracy of work in a machine shop	$\frac{1}{1000}$ inch
Accuracy in most refined measurements	$\frac{1}{10,000,000}$ inch
Dimensions of double tennis court	$78' \times 36'$
Dimensions of single tennis court	$78' \times 27'$
Dimensions of football field	$160' \times 330'$
Standard width of railroad track	$4'\ 8\frac{1}{2}''$

Weights and Measures

Avoirdupois weight, 1 ton = 2000 lb.; 1 lb. = 16 oz. = 7000 gr.

Troy weight, 1 lb. = 12 oz. = 5760 gr.; 1 oz. = 20 pwt. = 480 gr.

Long measure, 1 mi. = 1760 yd. = 5280 ft. = 63,360 in.

Square measure, 1 A. = 160 sq. rd. = 43,560 sq. ft.; 1 sq. yd. = 9 sq. ft. = 9 × 144 sq. in.

Cubic measure, 1 cu. yd. = 27 cu. ft. = 27 × 1728 cu. in.

Dry measure, 1 bu. = 4 pk. = 32 qt. = 64 pt.

Liquid measure, 1 gal. = 4 qt. = 8 pt.; 1 pt. = 16 liquid oz.

Paper measure, 1 ream = 20 quires = 480 sheets.

Metric system, 1 meter = 39.37 in.; 1 kilometer = .621 mi.
1 liter = 1.057 liquid qt.; 1 kilogram = 2.2046 lb.
1 hectare = 2.471 A.
1 kilometer = 10 hectometers = 100 decameters = 1000 meters = 10,000 decimeters = 100,000 centimeters = 1,000,000 millimeters.

At the option of the teacher, the pupil may insert on the blank pages at the end of the book other important formulas or numerical facts, particularly those which are important in the locality in which the pupil lives.

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